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T-Spherical Fuzzy Graphs: Operations and Applications in Various Selection Processes

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Abstract

The notion of T-spherical fuzzy set is the most recent generalization of intuitionistic fuzzy set available in the literature having the capability to handle the uncertainty, fuzziness and vagueness found in human sense in terms of the four parameters: membership (yes), neutral (abstain), non-membership (no) and refusal (non-participation). In the present communication, the concept of T-spherical fuzzy graph has been introduced along with the operations of product, composition, union, join and complement. Further, two algorithms utilizing the notion of T-spherical fuzzy graphs have been presented for solving the decision-making problems in the field of supply chain management and evaluation problem of service centers. In order to illustrate the actual implementation of the proposed algorithms, numerical examples have also been provided. For the sake of the novelty of the proposed approach, comparison and advantages in contrast with the methodologies of intuitionistic fuzzy set and Pythagorean fuzzy set have also been discussed.

Keywords T-spherical fuzzy set · Spherical fuzzy preference relation · Intuitionistic fuzzy graphs · Score function · Decision making

1 Introduction

For the sake of wider applicability and more coverage of human flexibility, the researchers have presented various generalizations for the fuzzy sets (FSs) [1] and the intuitionistic fuzzy sets (IFSs) [2] to model the uncertainties and the hesitancy inherent in many practical circumstances. Yager [3] revealed that the existing structures of FS and IFS are not capable enough to depict the human opinion in more practical/broader sense and introduced the notion of Pythagorean fuzzy set (PyFS) which effectively enlarged the span of information by introducing the new conditional constraint, where the squared sum of membership and non-membership is ≤ 1. Thus, the concept of membership/belongingness (yes), non-membership/non-belongingness (no) and indeterminacy/neutral (abstain) has differently been taken into account in the respective definitions of IFS and PyFS.

Liu et al. [4] studied a new extension of linguistic term called as Pythagorean uncertain linguistic sets along with its various operators and proposed a Pythagorean uncertain linguistic partitioned Bonferroni mean operator and its weighted form which resulted in a new methodology to solve a multi-attribute decision-making problem. Teng et al. [5] proposed a power Maclaurin symmetric mean operator and its weighted form with their application in decision making. Further, Liu et al. studied various operators of q-rung orthopair fuzzy sets to solve the multi-attribute decision-making problem, viz aggregation operators [6], power Maclaurin symmetric mean operator with its weighted form [7] and Bonferroni mean operator & its weighted form [8]. Recently, based on Archimedean t-norm and t-conorm, Liu and Wang [9] presented q-rung orthopair fuzzy Archimedean Bonferroni operators with its weighted form and proposed a new methodology to solve a multi-attribute decision-making problem.

In the literature, we see an example of a voting system where the voters have been categorized into four different classes—one who votes for (yes), one who votes against (no), one who neither vote for nor against (abstain) and one who refused for voting (refusal). It may be noted that the concept of "refusal" is found to be an additional component which was not being taken into account by any of the sets or by their generalizations stated above. In order to deal with

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such circumstances and to develop a formal concept which would be sufficiently close to cater the humans nature of flexibility, Cuong [10] introduced the notion of picture fuzzy set (PFS) in which all the four parameters, i.e., degree of membership, degree of indeterminacy (neutral), degree of non-membership and the degree of refusal have been taken into account.

In order to have a further extension, recently Mahmood et al. [11] introduced the notion of spherical fuzzy set (SFS) and T-spherical fuzzy set (TSFS) which give an additional strength to the idea of picture fuzzy sets by broadening/enlarging the space for the grades of all the four parameters. There are some real-world problems (e.g., the voting system stated above) where the information cannot be represented adequately by using the Pythagorean fuzzy graphs/sets. So to capture the information content and utilize the flexibility, we extend the literature by using T-spherical fuzzy set and investigate in its various applications. Figure 1 briefly demonstrates the geometrical prospects of various generalizations of fuzzy sets based on different constraint conditions.

Kifayat et al. [12] studied the geometrical comparison of FSs, IFSs, PyFSs, PFSs with SFSs and TSFSs. Also, they proposed and studied various similarity measures for intuitionistic fuzzy sets and picture fuzzy sets which have their own limitations and could not be applied in the broader setup as of spherical fuzzy environment. Further, they proposed various types of similarity measures for TSFSs with their useful applications in various fields, e.g., decision making, medical diagnosis and pattern recognition. Next, Liu et al. [13] developed a T-spherical fuzzy power Muirhead operator and devised an algorithm to solve a multi-attribute decision-making problem.

The responsibility as well as socioeconomic risk factors and financial implications in the business increases with the increasing awareness of customer's knowledge on sustainability and environmental operations. Therefore, the choice of supplier is one of the major tasks in the process of supply chain management. It may be observed that the supplier selection process mainly consists of four steps—classifying suppliers, gathering supplier information, to deal with suppliers, assessment of suppliers and then choosing the suitable one. In reference with the multi-criteria decision-making approach [14,15], the process of assimilating the information about the suppliers means identifying the attributes which are required to be assessed. Dealing with the suppliers means determining the attribute's values of every supplier. The major task in the assessment of the suppliers is to sort them and select the best one based on the values of these attributes.

In many practical situations such as operation management, networking, system analysis, economical interpretation and decision support system, the graph-theoretic representations of the information have been found to be more effective and convenient to deal with the information embedded among different objects/attributes/alternatives. Based on the fuzzy relation [16], Kaufmann [17] proposed the concept of fuzzy graphs and subsequently Rosenfeld [18] developed the concept of fuzzy vertex and fuzzy edge. Some standard operations on the fuzzy graphs were studied by the Mordeson and Peng [19] with their properties. Further, Parvathi et al. [20,21] extended the notion of a fuzzy graph to intuitionistic fuzzy graph and analyzed various properties related to minmax intuitionistic fuzzy graph. Karunambigai et al. [22] proposed a category of constant and totally constant intuitionistic fuzzy graphs, and subsequently Akram and Davvaz [23] presented the concept of strong intuitionistic fuzzy graphs along with their properties. Also, Akram and Dudek [24] presented intuitionistic fuzzy hypergraphs with their applications, and Alshehri and Akram [25] defined the planarity, duality and multigraphs in context with intuitionistic fuzzy graphs. Sahoo and Pal [26,27] proposed various types of product operations for intuitionistic fuzzy graphs, intuitionistic fuzzy tolerance graph with their applications. Smarandache [28] defined the neutrosophic set (NS) by introducing the degree of indeterminacy as an independent component. Further, Quek et al. [29] extended the existing literature by introducing complex neutrosophic graphs of type 1 with utility in multi-attribute decision-making problem related to internet server selection.

Various researchers [30–33] utilized the flexibility of intuitionistic fuzzy graphs and applied to set some new ideas in concern with the extended structures of intuitionistic fuzzy graphs and provided many interesting applications in the field of clustering, decision-making problems and support systems. Recently, Naz et al. [34] proposed a generalization of the intuitionistic fuzzy graph termed as the Pythagorean fuzzy graphs and studied their applications in various decision-making problems. Some graph-theoretic operations related with Pythagorean fuzzy graphs have been well studied by Verma et al. [35].

Based on the above discussions, we have the following observations:

- Spherical fuzzy set is better enough than intuitionistic fuzzy set, Pythagorean fuzzy set and picture fuzzy set to express the fuzzy information/vagueness.
- The graph-theoretic representations of information are more effective and convenient to deal with the information embedded among different attributes/alternatives. However, in the available literature, the graph representation has not been applied with the spherical fuzzy sets.
- Therefore, the expressions consisting of spherical fuzzy set notion and its graph-theoretic representations are supposed to be more flexible and having broader span of information coverage to deal with the decision-making problems.

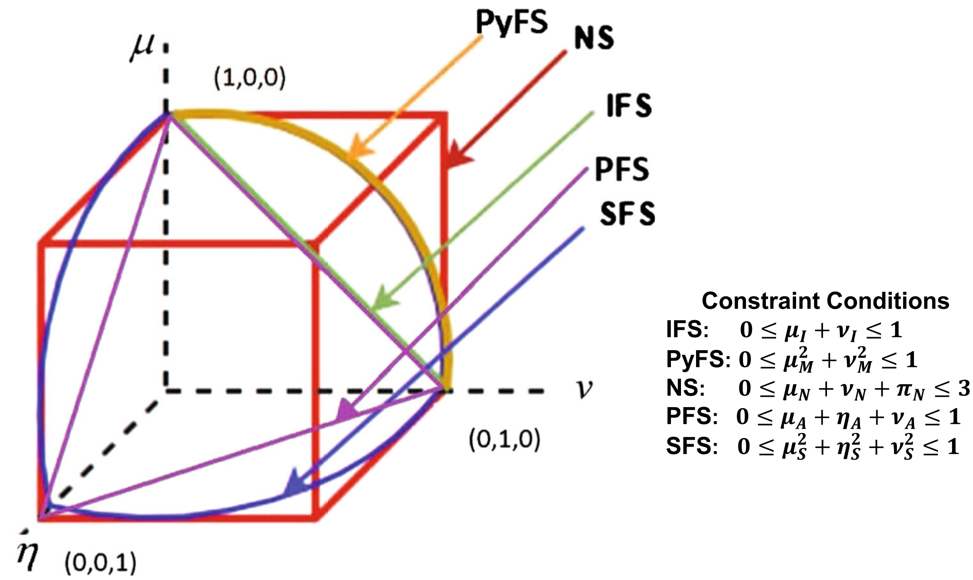

Fig. 1 Geometrical representations of generalizations of fuzzy set

Thus, the objective of the paper is to formally enhance the graph-theoretic notions under spherical fuzzy environment for the sake of wider span and broader coverage of the information. We propose a new category of graph called T-spherical fuzzy graph and study various aspects of it along with its applications.

The work in the present manuscript has been organized as follows. In Sect. 2, we present some basic definitions and preliminaries related to the spherical fuzzy sets, T-spherical fuzzy sets, intuitionistic fuzzy graph and Pythagorean fuzzy graph. Considering the fact that T-spherical fuzzy sets have the immense capability to model the imprecise, vague, uncertain or incomplete information inherent in the real-world applications, a new kind of T-spherical fuzzy graph has been defined in Sect. 3, and various operations over these graphs have been studied in Sect. 4. Further, we present two algorithms to solve two different types of decision-making problems by utilizing the notion of T-spherical fuzzy graph in Sect. 5. Illustrative examples have also been discussed to demonstrate the implementation of the proposed algorithms. Comparative study and advantages with IFS/PyFS have been provided in brief in Sect. 6. Finally, the paper has been concluded in Sect. 7 indicating the scope for future work.

2 Preliminaries

In this section, we recall and present some basic definitions of the various generalizations of fuzzy sets, such as IFS, PyFS, PFS, SFS, TSFS, intuitionistic fuzzy graph and Pythagorean fuzzy graph.

Definition 1 [2] An intuitionistic fuzzy set I in U (universe of discourse) is given by:

$$I = \{< \alpha, \mu_I(\alpha), \nu_I(\alpha) >| \alpha \in U\};$$

where $\mu_I: U \to [0, 1]$ and $\nu_I: U \to [0, 1]$ denote the degree of membership and degree of non-membership, respectively, and for every $\alpha \in U$ satisfy the condition:

$$0 \le \mu_I(\alpha) + \nu_I(\alpha) \le 1.$$

The degree of indeterminacy is given by: $\pi_I(\alpha) = 1 - \mu_I(\alpha) - \nu_I(\alpha); \forall \alpha \in U$.

Definition 2 [3] A Pythagorean fuzzy set M in U(universe of discourse) is given by:

$$M = \{< \alpha, \mu_M(\alpha), \nu_M(\alpha) >| \alpha \in U\};$$

where $\mu_M: U \to [0, 1]$ and $\nu_M: U \to [0, 1]$ denote the degree of membership and degree of non-membership, respectively, and for every $\alpha \in U$ satisfy the condition:

$$0 \le \mu_M^2(\alpha) + \nu_M^2(\alpha) \le 1.$$

The degree of indeterminacy for any Pythagorean fuzzy set M and $\alpha \in U$ is given by:

$$\pi_M(\alpha) = \sqrt{1 - \mu_M^2(\alpha) - \nu_M^2(\alpha)}.$$

Definition 3 [10] A picture fuzzy set A in U (universe of discourse) is given by:

$$A = \{< \alpha, \mu_A(\alpha), \eta_A(\alpha), \nu_A(\alpha) >| \alpha \in U\};$$

where $\mu_A: U \to [0,1]$, $\eta_A: U \to [0,1]$ and $\nu_A: U \to [0,1]$ denote the degree of membership, degree of neutral membership (abstain) and degree of non-membership, respectively, and for every $\alpha \in U$ satisfy the condition:

$$\mu_A(\alpha) + \eta_A(\alpha) + \nu_A(\alpha) \leq 1.$$

The degree of refusal for any picture fuzzy set A and $\alpha \in U$ is given by:

$$r_A(\alpha) = 1 - (\mu_A(\alpha) + \eta_A(\alpha) + \nu_A(\alpha)).$$

Definition 4 [11] A spherical fuzzy set S in U (universe of discourse) is given by:

$$S = \{< \alpha, \mu_S(\alpha), \eta_S(\alpha), \nu_S(\alpha) > | \alpha \in U\};$$

where $\mu_S: U \to [0,1]$, $\eta_S: U \to [0,1]$ and $\nu_S: U \to [0,1]$ denote the degree of membership, degree of neutral membership (abstain) and degree of non-membership, respectively, and for every $\alpha \in U$ satisfy the condition:

$$\mu_S^2(\alpha) + \eta_S^2(\alpha) + \nu_S^2(\alpha) \leq 1, \quad \forall \alpha \in U.$$

The degree of refusal for any spherical fuzzy set S and $\alpha \in U$ is given by:

$$r_S(\alpha) = \sqrt{1 - \left(\mu_S^2(\alpha) + \eta_S^2(\alpha) + \nu_S^2(\alpha)\right)}.$$

Definition 5 [11] A T-spherical fuzzy set S in U (universe of discourse) is given by:

$$S = \{< \alpha, \mu_S(\alpha), \eta_S(\alpha), \nu_S(\alpha) > | \alpha \in U\};$$

where $\mu_S: U \to [0,1]$, $\eta_S: U \to [0,1]$ and $\nu_S: U \to [0,1]$ denote the degree of membership, degree of neutral membership (abstain) and degree of non-membership, respectively, and for every $\alpha \in U$ satisfy the condition:

$$\mu_S^n(\alpha) + \eta_S^n(\alpha) + \nu_S^n(\alpha) \leq 1, \quad \forall\ \alpha \in U.$$

The degree of refusal for any T-spherical fuzzy set S and $\alpha \in U$ is given by:

$$r_S(\alpha) = \sqrt[n]{1 - (\mu_S^n(\alpha) + \eta_S^n(\alpha) + \nu_S^n(\alpha))}.$$

Particular Cases:

- For $n = 2$, T-spherical fuzzy set reduces to spherical fuzzy set.
- For $n = 1$, T-spherical fuzzy set reduces to picture fuzzy set.
- If $n = 2$ & $r_S = 0$, then T-spherical fuzzy set reduces to Pythagorean fuzzy set.
- If $n = 1$ & $r_S = 0$, then T-spherical fuzzy set reduces to intuitionistic fuzzy set.

Definition 6 Let U be a universal set. An intuitionistic fuzzy graph [20] on U is denoted by $\widetilde{G} = (P, Q)$, where P is an intuitionistic fuzzy set on U and Q is an intuitionistic fuzzy relation in $U \times U$ such that

$$\mu_Q(\alpha, \beta) \leq \min\{\mu_P(\alpha), \mu_P(\beta)\},$$
$$\nu_Q(\alpha, \beta) \geq \max\{\nu_P(\alpha), \nu_P(\beta)\},$$

satisfying the constraint condition

$$0 \leq \mu_Q^2(\alpha, \beta) + \nu_Q^2(\alpha, \beta) \leq 1, \ \forall \alpha, \beta \in U.$$

The set P is called the intuitionistic fuzzy vertex set of the graph $\widetilde{G}$, and Q is called the intuitionistic fuzzy edge set of the graph $\widetilde{G}$.

Definition 7 A Pythagorean fuzzy graph [34] on U is denoted by $\widehat{G} = (M, N)$, where M is a Pythagorean fuzzy set on U and N is a Pythagorean fuzzy relation in $U \times U$ such that

$$\mu_N(\alpha, \beta) \leq \min\{\mu_M(\alpha), \mu_M(\beta)\},$$
$$\nu_N(\alpha, \beta) \geq \max\{\nu_M(\alpha), \nu_M(\beta)\},$$

satisfying the constraint condition $0 \leq \mu_N^2(\alpha, \beta) + \nu_N^2(\alpha, \beta) \leq 1, \ \forall \alpha, \beta \in U$. The set M is called the Pythagorean fuzzy vertex set of the graph $\widehat{G}$, and N is called the Pythagorean fuzzy edge set of the graph $\widehat{G}$.

3 *T*-Spherical Fuzzy Graphs and Relations

In this section, we propose the notion of T-spherical fuzzy graph as a new category of the graph associated with T-spherical fuzzy set (TSFS) by introducing the definition of T-spherical fuzzy relation (TSFR) as follows:

Definition 8 Let U be a universal set. A T-*spherical fuzzy relation* in U is a T-spherical fuzzy set R in $U \times U$, given by:

$$R = \{< (\alpha, \beta), \mu_R(\alpha, \beta), \eta_R(\alpha, \beta), \nu_R(\alpha, \beta) > |(\alpha, \beta) \in U \times U\},$$

where

$$\mu_R: U \times U \to [0,1], \ \ \eta_R: U \times U \to [0,1] \text{ and }$$
$$\nu_R: U \times U \to [0,1]$$

represent the degree of membership, degree of neutral membership (abstain) and degree of non-membership, respectively, satisfying the condition

$$\mu_R^n(\alpha,\beta)+\eta_R^n(\alpha,\beta)+\nu_R^n(\alpha,\beta)\le 1;\quad \forall(\alpha,\beta)\in U\times U.$$

Definition 9 Let U be a universal set. A T-spherical fuzzy relation R in U is said to be a *symmetric* T-spherical fuzzy relation if

$$\mu_R(\alpha,\beta)=\mu_R(\beta,\alpha),\ \eta_R(\alpha,\beta)=\eta_R(\beta,\alpha),$$
$$\nu_R(\alpha,\beta)=\nu_R(\beta,\alpha)\ \forall\alpha,\beta\in U.$$

Let $R=(\mu_R,\eta_R,\nu_R)$ and $S=(\mu_S,\eta_S,\nu_S)$ be two T-spherical fuzzy sets defined on U. Further, suppose that R is a T-spherical fuzzy relation on U. Then, R is called T-spherical fuzzy relation on S if

$$\mu_R(\alpha,\beta)\le\min\{\mu_S(\alpha),\mu_S(\beta)\};$$
$$\eta_R(\alpha,\beta)\le\min\{\eta_S(\alpha),\eta_S(\beta)\};$$
$$\nu_R(\alpha,\beta)\le\max\{\nu_S(\alpha),\nu_S(\beta)\},$$

for all $\alpha,\ \beta\in U$ and satisfying the condition $0\le\mu_R^2(\alpha,\beta)+\eta_R^2(\alpha,\beta)+\nu_R^2(\alpha,\beta)\le 1$.

Definition 10 Let U be a universal set. A T-*spherical fuzzy graph* on U is denoted by $G=(S,R)$, where S is a TSFS on U with $\mu_S^n(\alpha)+\eta_S^n(\alpha)+\nu_S^n(\alpha)\le 1,\quad\forall\alpha\in U$ and R is a TSFR in $U\times U$ such that

$$\mu_R(\alpha,\beta)\le\min\{\mu_S(\alpha),\mu_S(\beta)\},$$
$$\eta_R(\alpha,\beta)\le\min\{\eta_S(\alpha),\eta_S(\beta)\},$$
$$\nu_R(\alpha,\beta)\le\max\{\nu_S(\alpha),\nu_S(\beta)\},$$

and satisfying the condition

$$\mu_R^n(\alpha,\beta)+\eta_R^n(\alpha,\beta)+\nu_R^n(\alpha,\beta)\le 1\ \ \forall\alpha,\beta\in U.$$

Here, S and R are the T-spherical fuzzy vertex set and T-spherical fuzzy edge set of the T-spherical fuzzy graph G, respectively.

Remarks (i) If $n=2$, then the T-spherical fuzzy graph reduces to spherical fuzzy graph.
(ii) In the above definition, if R is a symmetric relation on S, then $G=(S,R)$ is called T-spherical fuzzy graph.
(iii) If R is not a symmetric relation on S, then $G=(S,R)$ is called T-spherical fuzzy directed graph.
(iv) If $\mu_R(\alpha,\beta)=0,\ \eta_R(\alpha,\beta)=0,\ \nu_R(\alpha,\beta)=0$ for some $\alpha\ \&\ \beta\in U$, then there is no edge between α and β.
(v) If there is an edge between α and β, then one of the following conditions must be satisfied:

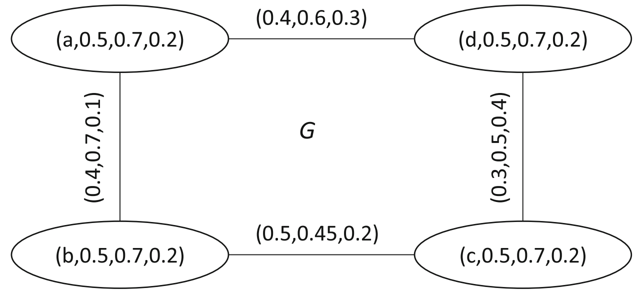

Fig. 2 Graph $G=(S,R)$

- $\mu_R(\alpha,\beta)=0,\ \eta_R(\alpha,\beta)=0,\ \nu_R(\alpha,\beta)>0;$
- $\mu_R(\alpha,\beta)=0,\ \eta_R(\alpha,\beta)>0,\ \nu_R(\alpha,\beta)=0;$
- $\mu_R(\alpha,\beta)>0,\ \eta_R(\alpha,\beta)=0,\ \nu_R(\alpha,\beta)=0;$
- $\mu_R(\alpha,\beta)=0,\ \eta_R(\alpha,\beta)>0,\ \nu_R(\alpha,\beta)>0;$
- $\mu_R(\alpha,\beta)>0,\ \eta_R(\alpha,\beta)=0,\ \nu_R(\alpha,\beta)>0;$
- $\mu_R(\alpha,\beta)>0,\ \eta_R(\alpha,\beta)>0,\ \nu_R(\alpha,\beta)=0;$
- $\mu_R(\alpha,\beta)>0,\ \eta_R(\alpha,\beta)>0,\ \nu_R(\alpha,\beta)>0.$

Example 1 Let $G^{'}=(V,E)$ be a graph such that $V=\{a,b,c,d\}$ and $E=\{(a,b),(b,c),(c,d),(d,a)\}\subseteq V\times V$. Consider S to be a T-spherical fuzzy vertex set in V given by $S=\{(a,0.5,0.7,0.2),(b,0.8,0.3,0.1),(c,0.6,0.5,0.2),(d,0.4,0.4,0.5)\}$ and R be a T-spherical fuzzy edge set in E given by:

$$R=\{((a,b),0.4,0.7,0.1),((b,c),0.5,0.45,0.2),$$
$$((c,d),0.3,0.5,0.4),((d,a),0.4,0.6,0.3)\}.$$

We represent $G=(S,R)$ as a T-spherical fuzzy graph of $G^{'}$ in Fig. 2.

Definition 11 Let $G^{'}=(V,E)$ be a graph such that $V=\{a,b,c,d\}$ and $E=\{(a,b),(b,c),(c,d),(d,a)\}\subseteq V\times V$. Consider S be the T-spherical fuzzy vertex set in V and R be the T-spherical fuzzy edge set in E. Then, we define the *degree* and the *total degree* of a vertex $a\in V$ for the T-spherical fuzzy graph as follows:

$$\deg_G(a)=\left(d_\mu(a),d_\eta(a),d_\nu(a)\right)$$

and

$$\mathrm{Tdeg}_G(a)=\left(td_\mu(a),td_\eta(a),td_\nu(a)\right),$$

respectively, where

$$d_\mu(a)=\sum_{a,b\ne a\in V}\mu_R(a,b),$$
$$d_\eta(a)=\sum_{a,b\ne a\in V}\eta_R(a,b),$$

$$d_\nu(a) = \sum_{a,b\neq a\in V} \nu_R(a,b);$$
$$td_\mu(a) = \sum_{a,b\neq a\in V} \mu_R(a,b) + \mu_S,$$
$$d_\eta(a) = \sum_{a,b\neq a\in V} \eta_R(a,b) + \eta_S,$$
$$d_\nu(a) = \sum_{a,b\neq a\in V} \nu_R(a,b) + \nu_S.$$

4 Operations on *T*-spherical Fuzzy Graphs

In this section, we propose some important graph-theoretic operations over T-spherical fuzzy graphs along with various important results and illustrative examples.

Let $G_1 = (S_1, R_1)$ and $G_2 = (S_2, R_2)$ be two T-spherical fuzzy graphs with reference to the graphs $G^{'} = (V_1, E_1)$ and $G^{''} = (V_2, E_2)$, respectively, where S_1 & S_2 are the T-spherical fuzzy vertex sets in V_1 & V_2, respectively, and R_1 & R_2 are the T-spherical fuzzy edge sets in E_1 & E_2, respectively.

- *Cartesian Product of T-spherical Fuzzy Graph*
  The Cartesian product of two T-spherical fuzzy graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined as follows:

  $$G_1 \times G_2 = (S_1 \times S_2, R_1 \times R_2),$$

  where
- $\mu_{S_1\times S_2}(\alpha_1,\alpha_2) = \min(\mu_{S_1}(\alpha_1), \mu_{S_2}(\alpha_2))$,
  $\eta_{S_1\times S_2}(\alpha_1,\alpha_2) = \min(\eta_{S_1}(\alpha_1), \eta_{S_2}(\alpha_2))$,
  $\nu_{S_1\times S_2}(\alpha_1,\alpha_2) = \max(\nu_{S_1}(\alpha_1), \nu_{S_2}(\alpha_2))$, $\forall(\alpha_1,\alpha_2) \in V_1 \times V_2$;
- $\mu_{R_1\times R_2}((\alpha,\alpha_2),(\alpha,\beta_2)) = \min(\mu_{S_1}(\alpha), \mu_{R_2}(\alpha_2,\beta_2))$,
  $\eta_{R_1\times R_2}((\alpha,\alpha_2),(\alpha,\beta_2)) = \min(\eta_{S_1}(\alpha), \eta_{R_2}(\alpha_2,\beta_2))$,
  $\nu_{R_1\times R_2}((\alpha,\alpha_2),(\alpha,\beta_2)) = \max(\nu_{S_1}(\alpha), \nu_{R_2}(\alpha_2,\beta_2))$,
  $\forall\alpha \in V_1, (\alpha_2,\beta_2) \in E_2$;
- $\mu_{R_1\times R_2}((\alpha_1,\gamma),(\beta_1,\gamma)) = \min(\mu_{R_1}(\alpha_1,\beta_1), \mu_{S_2}(\gamma))$,
  $\eta_{R_1\times R_2}((\alpha_1,\gamma),(\beta_1,\gamma)) = \min(\eta_{R_1}(\alpha_1,\beta_1), \eta_{S_2}(\gamma))$,
  $\nu_{R_1\times R_2}((\alpha_1,\gamma),(\beta_1,\gamma)) = \max(\nu_{R_1}(\alpha_1,\beta_1), \nu_{S_2}(\gamma))$,
  $\forall\gamma \in V_2, (\alpha_1,\beta_1) \in E_1$.

Theorem 1 *The Cartesian product of two T-spherical fuzzy graphs is a T-spherical fuzzy graph.*

Proof Suppose $\alpha \in V_1$ and $(\alpha_2,\beta_2) \in E_2$. Then,

$$\begin{aligned}\mu_{R_1\times R_2}((\alpha,\alpha_2),(\alpha,\beta_2)) &= \min\left(\mu_{S_1}(\alpha), \mu_{R_2}(\alpha_2,\beta_2)\right),\\ &\le \min\left(\mu_{S_1}(\alpha), \min\left(\mu_{S_2}(\alpha_2), \mu_{S_2}(\beta_2)\right)\right),\\ &= \min\left(\min\left(\mu_{S_1}(\alpha), \mu_{S_2}(\alpha_2)\right), \min\left(\mu_{S_1}(\alpha), \mu_{S_2}(\beta_2)\right)\right),\\ &= \min\left(\mu_{S_1\times S_2}(\alpha,\alpha_2), \mu_{S_1\times S_2}(\alpha,\beta_2)\right);\end{aligned}$$

$$\begin{aligned}\eta_{R_1\times R_2}((\alpha,\alpha_2),(\alpha,\beta_2)) &= \min(\eta_{S_1}(\alpha), \eta_{R_2}(\alpha_2,\beta_2)),\\ &\le \min\left(\eta_{S_1}(\alpha), \min\left(\eta_{S_2}(\alpha_2), \eta_{S_2}(\beta_2)\right)\right),\\ &= \min\left(\min\left(\eta_{S_1}(\alpha), \eta_{S_2}(\alpha_2)\right), \min\left(\eta_{S_1}(\alpha), \eta_{S_2}(\beta_2)\right)\right),\\ &= \min\left(\eta_{S_1\times S_2}(\alpha,\alpha_2), \eta_{S_1\times S_2}(\alpha,\beta_2)\right);\end{aligned}$$

and

$$\begin{aligned}\nu_{R_1\times R_2}((\alpha,\alpha_2),(\alpha,\beta_2)) &= \max\left(\nu_{S_1}(\alpha), \nu_{R_2}(\alpha_2,\beta_2)\right),\\ &\le \max\left(\nu_{S_1}(\alpha), \max\left(\nu_{S_2}(\alpha_2), \nu_{S_2}(\beta_2)\right)\right),\\ &= \max\left(\max\left(\nu_{S_1}(\alpha), \nu_{S_2}(\alpha_2)\right), \max\left(\nu_{S_1}(\alpha), \nu_{S_2}(\beta_2)\right)\right),\\ &= \max\left(\nu_{S_1\times S_2}(\alpha,\alpha_2), \nu_{S_1\times S_2}(\alpha,\beta_2)\right).\end{aligned}$$

Similarly, if we consider $\gamma \in V_2$, $(\alpha_1,\beta_1) \in E_1$, then we have

$$\begin{aligned}\mu_{R_1\times R_2}((\alpha_1,\gamma),(\beta_1,\gamma)) &= \min\left(\mu_{R_1}(\alpha_1,\beta_1), \mu_{S_2}(\gamma)\right),\\ &\le \min\left(\min\left(\mu_{S_1}(\alpha_1), \mu_{S_1}(\beta_1), \mu_{S_2}(\gamma)\right)\right),\\ &= \min\left(\min\left(\mu_{S_1}(\alpha_1), \mu_{S_2}(\gamma)\right), \min\left(\mu_{S_1}(\beta_1), \mu_{S_2}(\gamma)\right)\right),\\ &= \min\left(\mu_{S_1\times S_2}(\alpha_1,\gamma), \mu_{S_1\times S_2}(\beta_1,\gamma)\right);\end{aligned}$$

$$\begin{aligned}\eta_{R_1\times R_2}((\alpha_1,\gamma),(\beta_1,\gamma)) &= \min\left(\eta_{R_1}(\alpha_1,\beta_1), \eta_{S_2}(\gamma)\right),\\ &\le \min\left(\min\left(\eta_{S_1}(\alpha_1, \eta_{S_1}(\beta_1)), \eta_{S_2}(\gamma)\right)\right),\\ &= \min\left(\min\left(\eta_{S_1}(\alpha_1), \eta_{S_2}(\gamma)\right), \min\left(\eta_{S_1}(\beta_1), \eta_{S_2}(\gamma)\right)\right),\\ &= \min\left(\eta_{S_1\times S_2}(\alpha_1,\gamma), \eta_{S_1\times S_2}(\beta_1,\gamma)\right);\end{aligned}$$

and

$$\begin{aligned}\nu_{R_1\times R_2}((\alpha_1,\gamma),(\beta_1,\gamma)) &= \max\left(\nu_{R_1}(\alpha_1,\beta_1), \nu_{S_2}(\gamma)\right),\\ &\le \max\left(\max\left(\nu_{S_1}(\alpha_1, \nu_{S_1}(\beta_1)), \nu_{S_2}(\gamma)\right)\right),\\ &= \max\left(\max\left(\nu_{S_1}(\alpha_1), \nu_{S_2}(\gamma)\right), \max\left(\nu_{S_1}(\beta_1), \nu_{S_2}(\gamma)\right)\right),\\ &= \max\left(\nu_{S_1\times S_2}(\alpha_1,\gamma), \nu_{S_1\times S_2}(\beta_1,\gamma)\right).\end{aligned}$$

Thus, in view of the definition of the T-spherical fuzzy graph, the result follows. The following example illustrates the above defined graph-theoretic operation. □

Example 2 Let $G_1^{'} = (V_1, E_1)$ and $G_2^{''} = (V_2, E_2)$ be two graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{(a, b)\}$ and $E_2 = \{(c, d)\}$. Let $G_1 = (S_1, R_1)$ and $G_2 = (S_2, R_2)$ be two T-spherical fuzzy graphs in reference with $G_1^{'}$ & $G_2^{''}$, respectively, where

$$S_1 = \{(a, 0.6, 0.2, 0.3), (b, 0.5, 0.1, 0.7)\},$$
$$R_1 = \{((a, b), 0.5, 0.2, 0.7)\};$$
$$S_2 = \{(c, 0.7, 0.1, 0.5), (d, 0.5, 0.2, 0.8)\},$$
$$R_2 = \{((c, d), 0.4, 0.1, 0.65)\}.$$

The graphs G_1, G_2 and its Cartesian product $G_1 \times G_2$ are being graphically presented in Figure 3.

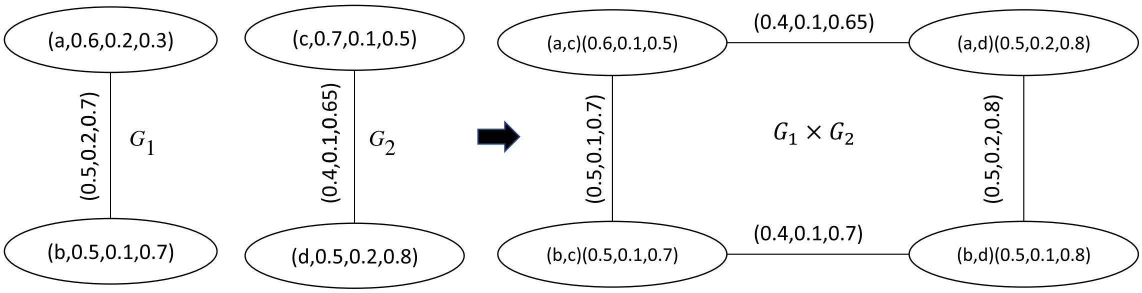

Fig. 3 Graphs G_1, G_2 and its Cartesian product $G_1 \times G_2$

– *Composition of T-spherical Fuzzy Graphs*
The composition of two T-spherical fuzzy graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is defined as follows:

$$G_1 \circ G_2 = (S_1 \circ S_2, R_1 \circ R_2),$$

where

- $\mu_{S_1 \circ S_2}(\alpha_1, \alpha_2) = \min(\mu_{S_1}(\alpha_1), \mu_{S_2}(\alpha_2))$, $\eta_{S_1 \circ S_2}(\alpha_1, \alpha_2) = \min(\eta_{S_1}(\alpha_1), \eta_{S_2}(\alpha_2))$, $\nu_{S_1 \circ S_2}(\alpha_1, \alpha_2) = \max(\nu_{S_1}(\alpha_1), \nu_{S_2}(\alpha_2))$, $\forall (\alpha_1, \alpha_2) \in V_1 \times V_2$;
- $\mu_{R_1 \circ R_2}((\xi, \alpha_2), (\xi, \beta_2)) = \min(\mu_{S_1}(\xi), \mu_{R_2}(\alpha_2, \beta_2))$, $\eta_{R_1 \circ R_2}((\xi, \alpha_2), (\xi, \beta_2)) = \min(\eta_{S_1}(\xi), \eta_{R_2}(\alpha_2, \beta_2))$, $\nu_{R_1 \circ R_2}((\xi, \alpha_2), (\xi, \beta_2)) = \max(\nu_{S_1}(\xi), \nu_{R_2}(\alpha_2, \beta_2))$, $\forall \xi \in V_1,\ (\alpha_2, \beta_2) \in E_2$;
- $\mu_{R_1 \circ R_2}((\alpha_1, \gamma), (\beta_1, \gamma)) = \min(\mu_{R_1}(\alpha_1, \beta_1), \mu_{S_2}(\gamma))$, $\eta_{R_1 \circ R_2}((\alpha_1, \gamma), (\beta_1, \gamma)) = \min(\eta_{R_1}(\alpha_1, \beta_1), \eta_{S_2}(\gamma))$, $\nu_{R_1 \circ R_2}((\alpha_1, \gamma), (\beta_1, \gamma)) = \max(\nu_{R_1}(\alpha_1, \beta_1), \nu_{S_2}(\gamma))$, $\forall \gamma \in V_2,\ (\alpha_1, \beta_1) \in E_1$;
- $\mu_{R_1 \circ R_2}((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \min(\mu_{S_2}(\alpha_2), \mu_{S_2}(\beta_2), \mu_{R_1}(\alpha_1, \beta_1))$,
  $\eta_{R_1 \circ R_2}((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \min(\eta_{S_2}(\alpha_2), \eta_{S_2}(\beta_2), \eta_{R_1}(\alpha_1, \beta_1))$,
  $\nu_{R_1 \circ R_2}((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \max(\nu_{S_2}(\alpha_2), \nu_{S_2}(\beta_2), \nu_{R_1}(\alpha_1, \beta_1))$,
  $((\alpha_1, \alpha_2), (\beta_1, \beta_2)) \in E^\circ$,
  where $E^\circ = \{((\alpha_1, \alpha_2), (\beta_1, \beta_2)) | (\alpha_1, \beta_1) \in E_1 \text{ and } \alpha_2 \neq \beta_2\}$.

Theorem 2 *The composition of two T-spherical fuzzy graphs is a T-spherical fuzzy graph.*

Proof Suppose $\xi \in V_1$ and $(\alpha_2, \beta_2) \in E_2$. Then,

$$\begin{aligned}
\mu_{R_1 \circ R_2}((\xi, \alpha_2), (\xi, \beta_2)) &= \min\left(\mu_{S_1}(\xi), \mu_{R_2}(\alpha_2, \beta_2)\right),\\
&\leq \min\left(\mu_{S_1}(\xi), \min\left(\mu_{S_2}(\alpha_2), \mu_{S_2}(\beta_2)\right)\right),\\
&= \min\left(\min\left(\mu_{S_1}(\xi), \mu_{S_2}(\alpha_2)\right), \min\left(\mu_{S_1}(\xi), \mu_{S_2}(\beta_2)\right)\right),\\
&= \min\left(\mu_{S_1 \circ S_2}(\xi, \alpha_2), \mu_{S_1 \circ S_2}(\xi, \beta_2)\right);
\end{aligned}$$

$$\begin{aligned}
\eta_{R_1 \circ R_2}((\xi, \alpha_2), (\xi, \beta_2)) &= \min\left(\eta_{S_1}(\xi), \eta_{R_2}(\alpha_2, \beta_2)\right),\\
&\leq \min\left(\eta_{S_1}(\xi), \min\left(\eta_{S_2}(\alpha_2), \eta_{S_2}(\beta_2)\right)\right),\\
&= \min\left(\min\left(\eta_{S_1}(\xi), \eta_{S_2}(\alpha_2)\right), \min\left(\eta_{S_1}(\xi), \eta_{S_2}(\beta_2)\right)\right),\\
&= \min\left(\eta_{S_1 \circ S_2}(\xi, \alpha_2), \eta_{S_1 \circ S_2}(\xi, \beta_2)\right);
\end{aligned}$$

and

$$\begin{aligned}
\nu_{R_1 \circ R_2}((\xi, \alpha_2), (\xi, \beta_2)) &= \max\left(\nu_{S_1}(\xi), \nu_{R_2}(\alpha_2, \beta_2)\right),\\
&\leq \max\left(\nu_{S_1}(\xi), \max\left(\nu_{S_2}(\alpha_2), \nu_{S_2}(\beta_2)\right)\right),\\
&= \max\left(\max\left(\nu_{S_1}(\xi), \nu_{S_2}(\alpha_2)\right), \max\left(\nu_{S_1}(\xi), \nu_{S_2}(\beta_2)\right)\right),\\
&= \max\left(\nu_{S_1 \circ S_2}(\xi, \alpha_2), \nu_{S_1 \circ S_2}(\xi, \beta_2)\right).
\end{aligned}$$

Similarly, if we consider $\gamma \in V_2$, $(\alpha_1, \beta_1) \in E_1$, then we have

$$\begin{aligned}
\mu_{R_1 \circ R_2}((\alpha_1, \gamma), (\beta_1, \gamma)) &= \min(\mu_{R_1}(\alpha_1, \beta_1), \mu_{S_2}(\gamma)),\\
&\leq \min\left(\min\left(\mu_{S_1}(\alpha_1), \mu_{S_1}(\beta_1)\right), \mu_{S_2}(\gamma)\right),\\
&= \min\left(\min\left(\mu_{S_1}(\alpha_1), \mu_{S_2}(\gamma)\right), \min\left(\mu_{S_1}(\beta_1), \mu_{S_2}(\gamma)\right)\right),\\
&= \min\left(\mu_{S_1 \circ S_2}(\alpha_1, \gamma), \mu_{S_1 \circ S_2}(\beta_1, \gamma)\right);
\end{aligned}$$

$$\begin{aligned}
\eta_{R_1 \circ R_2}((\alpha_1, \gamma), (\beta_1, \gamma)) &= \min(\eta_{R_1}(\alpha_1, \beta_1), \eta_{S_2}(\gamma)),\\
&\leq \min\left(\min\left(\eta_{S_1}(\alpha_1), \eta_{S_1}(\beta_1)\right), \eta_{S_2}(\gamma)\right),\\
&= \min\left(\min\left(\eta_{S_1}(\alpha_1), \eta_{S_2}(\gamma)\right), \min\left(\eta_{S_1}(\beta_1), \eta_{S_2}(\gamma)\right)\right),\\
&= \min\left(\eta_{S_1 \circ S_2}(\alpha_1, \gamma), \eta_{S_1 \circ S_2}(\beta_1, \gamma)\right);
\end{aligned}$$

and

$$\begin{aligned}
\nu_{R_1 \circ R_2}((\alpha_1, \gamma), (\beta_1, \gamma)) &= \max(\nu_{R_1}(\alpha_1, \beta_1), \nu_{S_2}(\gamma)),\\
&\leq \max\left(\max\left(\nu_{S_1}(\alpha_1), \nu_{S_1}(\beta_1)\right), \nu_{S_2}(\gamma)\right),
\end{aligned}$$

$$= \max\Big(\max\big(\nu_{S_1}(\alpha_1), \nu_{S_2}(\gamma)\big), \max\big(\nu_{S_1}(\beta_1), \nu_{S_2}(\gamma)\big)\Big),$$
$$= \max\big(\nu_{S_1 \circ S_2}(\alpha_1, \gamma), \nu_{S_1 \circ S_2}(\beta_1, \gamma)\big).$$

Further, if $\big((\alpha_1, \alpha_2), (\beta_1, \beta_2)\big) \in E^{\circ}$, $(\alpha_1, \beta_1) \in E_1$ and $\alpha_2 \neq \beta_2$, then we have,

$$\begin{aligned}
&\mu_{R_1 \circ R_2}\big((\alpha_1, \alpha_2), (\beta_1, \beta_2)\big)\\
&\quad = \min\big(\mu_{S_2}(\alpha_2), \mu_{S_2}(\beta_2), \mu_{R_1}(\alpha_1, \beta_1)\big),\\
&\quad \le \min\Big(\mu_{S_2}(\alpha_2), \mu_{S_2}(\beta_2), \min\big(\mu_{S_1}(\alpha_1), \mu_{S_1}(\beta_1)\big)\Big),\\
&\quad = \min\Big(\min\big(\mu_{S_2}(\alpha_2), \mu_{S_2}(\beta_2)\big), \min\big(\mu_{S_1}(\alpha_1), \mu_{S_1}(\beta_1)\big)\Big),\\
&\quad = \min\big(\mu_{S_1 \circ S_2}(\alpha_1, \alpha_2), \mu_{S_1 \circ S_2}(\beta_1, \beta_2)\big);\\
&\eta_{R_1 \circ R_2}\big((\alpha_1, \alpha_2), (\beta_1, \beta_2)\big)\\
&\quad = \min\big(\eta_{S_2}(\alpha_2), \eta_{S_2}(\beta_2), \eta_{R_1}(\alpha_1, \beta_1)\big),\\
&\quad \le \min\Big(\eta_{S_2}(\alpha_2), \eta_{S_2}(\beta_2), \min\big(\eta_{S_1}(\alpha_1), \eta_{S_1}(\beta_1)\big)\Big),\\
&\quad = \min\Big(\min\big(\eta_{S_2}(\alpha_2), \eta_{S_2}(\beta_2)\big), \min\big(\eta_{S_1}(\alpha_1), \eta_{S_1}(\beta_1)\big)\Big),\\
&\quad = \min\big(\eta_{S_1 \circ S_2}(\alpha_1, \alpha_2), \eta_{S_1 \circ S_2}(\beta_1, \beta_2)\big);
\end{aligned}$$

and

$$\begin{aligned}
&\nu_{R_1 \circ R_2}\big((\alpha_1, \alpha_2), (\beta_1, \beta_2)\big)\\
&\quad = \max\big(\nu_{S_2}(\alpha_2), \nu_{S_2}(\beta_2), \nu_{R_1}(\alpha_1, \beta_1)\big),\\
&\quad \le \max\Big(\nu_{S_2}(\alpha_2), \nu_{S_2}(\beta_2), \max\big(\nu_{S_1}(\alpha_1), \nu_{S_1}(\beta_1)\big)\Big),\\
&\quad = \max\Big(\max\big(\nu_{S_2}(\alpha_2), \nu_{S_2}(\beta_2)\big), \max\big(\nu_{S_1}(\alpha_1), \nu_{S_1}(\beta_1)\big)\Big),\\
&\quad = \max\big(\nu_{S_1 \circ S_2}(\alpha_1, \alpha_2), \nu_{S_1 \circ S_2}(\beta_1, \beta_2)\big).
\end{aligned}$$

Thus, in view of the definition of the T-spherical fuzzy graph, the result follows. The following example illustrates the above defined graph-theoretic operation. □

Example 3 Suppose $G_1^{'} = (V_1, E_1)$ and $G_2^{''} = (V_2, E_2)$ be two graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d\}$, $E_1 = \{(a, b)\}$ and $E_2 = \{(c, d)\}$. Let $G_1 = (S_1, R_1)$ and $G_2 = (S_2, R_2)$ be two T-spherical fuzzy graphs in reference with $G_1^{'}$ & $G_2^{''}$, respectively, where

$$\begin{aligned}
S_1 &= \{(a, 0.6, 0.2, 0.3), (b, 0.5, 0.1, 0.7)\},\\
R_1 &= \{((a, b), 0.5, 0.2, 0.7)\};\\
S_2 &= \{(c, 0.7, 0.1, 0.5), (d, 0.5, 0.2, 0.8)\},\\
R_2 &= \{((c, d), 0.4, 0.1, 0.65)\}.
\end{aligned}$$

Then, the graphs G_1, G_2 and their composition graph $G_1 \circ G_2$ are being graphically presented in Figure 4.

– *Union of T-spherical Fuzzy Graphs*
The union of two T-spherical fuzzy graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is defined as follows:

$$G_1 \cup G_2 = (S_1 \cup S_2, R_1 \cup R_2),$$

where

- $$\mu_{S_1 \cup S_2}(\alpha) = \begin{cases} \mu_{S_1}(\alpha) & \text{if } \alpha \in V_1 - V_2 \\ \mu_{S_2}(\alpha) & \text{if } \alpha \in V_2 - V_1 \\ \max\big(\mu_{S_1}(\alpha), \mu_{S_2}(\alpha)\big) & \text{if } \alpha \in V_1 \cup V_2 \end{cases}$$
$$\eta_{S_1 \cup S_2}(\alpha) = \begin{cases} \eta_{S_1}(\alpha) & \text{if } \alpha \in V_1 - V_2 \\ \eta_{S_2}(\alpha) & \text{if } \alpha \in V_2 - V_1 \\ \max\big(\eta_{S_1}(\alpha), \eta_{S_2}(\alpha)\big) & \text{if } \alpha \in V_1 \cup V_2 \end{cases}$$
$$\nu_{S_1 \cup S_2}(\alpha) = \begin{cases} \nu_{S_1}(\alpha) & \text{if } \alpha \in V_1 - V_2 \\ \nu_{S_2}(\alpha) & \text{if } \alpha \in V_2 - V_1 \\ \min\big(\nu_{S_1}(\alpha), \nu_{S_2}(\alpha)\big) & \text{if } \alpha \in V_1 \cup V_2 \end{cases}$$
- $$\mu_{R_1 \cup R_2}(\alpha, \beta) = \begin{cases} \mu_{R_1}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_1 - E_2 \\ \mu_{R_2}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_2 - E_1 \\ \max\big(\mu_{R_1}(\alpha, \beta), \mu_{R_2}(\alpha, \beta)\big) & \text{if } (\alpha, \beta) \in E_1 \cup E_2 \end{cases}$$
$$\eta_{R_1 \cup R_2}(\alpha, \beta) = \begin{cases} \eta_{R_1}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_1 - E_2 \\ \eta_{R_2}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_2 - E_1 \\ \max\big(\eta_{R_1}(\alpha, \beta), \eta_{R_2}(\alpha, \beta)\big) & \text{if } (\alpha, \beta) \in E_1 \cup E_2 \end{cases}$$
$$\nu_{R_1 \cup R_2}(\alpha, \beta) = \begin{cases} \nu_{R_1}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_1 - E_2 \\ \nu_{R_2}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_2 - E_1 \\ \min\big(\nu_{R_1}(\alpha, \beta), \nu_{R_2}(\alpha, \beta)\big) & \text{if } (\alpha, \beta) \in E_1 \cup E_2 \end{cases}$$

– *Join of T-spherical Fuzzy Graphs*
The join of two T-spherical fuzzy graphs G_1 and G_2, denoted by $G_1 + G_2$, is defined as follows:

$$G_1 + G_2 = (S_1 + S_2, R_1 + R_2),$$

where

- $$\mu_{S_1 + S_2}(\alpha) = \begin{cases} \mu_{S_1}(\alpha) & \text{if } \alpha \in V_1 - V_2 \\ \mu_{S_2}(\alpha) & \text{if } \alpha \in V_2 - V_1 \\ \mu_{S_1 \cup S_2}(\alpha) & \text{if } \alpha \in V_1 \cup V_2 \end{cases}$$
$$\eta_{S_1 + S_2}(\alpha) = \begin{cases} \eta_{S_1}(\alpha) & \text{if } \alpha \in V_1 - V_2 \\ \eta_{S_2}(\alpha) & \text{if } \alpha \in V_2 - V_1 \\ \eta_{S_1 \cup S_2}(\alpha) & \text{if } \alpha \in V_1 \cup V_2 \end{cases}$$

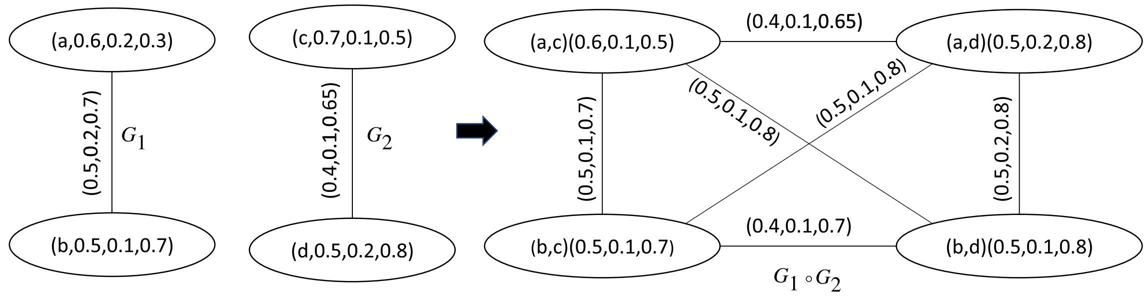

Fig. 4 Graphs G_1, G_2 and composition graph $G_1 \circ G_2$

$$\nu_{S_1+S_2}(\alpha) = \begin{cases} \nu_{S_1}(\alpha) & \text{if } \alpha \in V_1 - V_2 \\ \nu_{S_2}(\alpha) & \text{if } \alpha \in V_2 - V_1 \\ \nu_{S_1 \cup S_2}(\alpha) & \text{if } \alpha \in V_1 \cup V_2 \end{cases}$$

- $\mu_{R_1+R_2}(\alpha, \beta)$
$$= \begin{cases} \mu_{R_1}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_1 - E_2 \\ \mu_{R_2}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_2 - E_1 \\ \mu_{R_1 \cup R_2}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_1 \cup E_2 \end{cases}$$
$\eta_{R_1+R_2}(\alpha, \beta)$
$$= \begin{cases} \eta_{R_1}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_1 - E_2 \\ \eta_{R_2}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_2 - E_1 \\ \eta_{R_1 \cup R_2}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_1 \cup E_2 \end{cases}$$
$\nu_{R_1+R_2}(\alpha, \beta)$
$$= \begin{cases} \nu_{R_1}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_1 - E_2 \\ \nu_{R_2}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_2 - E_1 \\ \nu_{R_1 \cup R_2}(\alpha, \beta) & \text{if } (\alpha, \beta) \in E_1 \cup E_2 \end{cases}$$
- $\mu_{R_1+R_2}(\alpha, \beta) = \min\left(\mu_{S_1}(\alpha), \mu_{S_2}(\beta)\right) \quad \text{if} \quad (\alpha, \beta) \in E'$
$\eta_{R_1+R_2}(\alpha, \beta) = \min\left(\eta_{S_1}(\alpha), \eta_{S_2}(\beta)\right) \quad \text{if} \quad (\alpha, \beta) \in E'$
$\nu_{R_1+R_2}(\alpha, \beta) = \max\left(\nu_{S_1}(\alpha), \nu_{S_2}(\beta)\right) \quad \text{if} \quad (\alpha, \beta) \in E'$

where E' represents the set of all the edges joining the nodes of V_1 & V_2.

Theorem 3 *The union and join of two T-spherical fuzzy graphs are also T-spherical graphs.*

Proof The proof can be outlined similarly as the proof of Theorem 2. □

The operation of union and join is being illustrated with the help of the following examples:

Example 4 (Union of T-spherical Fuzzy Graphs) Suppose $G_1' = (V_1, E_1)$ and $G_2'' = (V_2, E_2)$ are two graphs such that $V_1 = \{a, b, c, d, e\}$, $V_2 = \{a, b, c, d, f\}$, $E_1 = \{(a, b), (b, c), (a, d), (d, e), (b, e), (c, e)\}$ and $E_2 = \{(a, b), (b, c), (b, d), (b, f), (c, f)\}$. Let $G_1 = (S_1, R_1)$ and $G_2 = (S_2, R_2)$ be two T-spherical fuzzy graphs in reference with G_1' & G_2'', respectively, where

$$S_1 = \{(a, 0.3, 0.1, 0.8), (b, 0.5, 0.2, 0.6), (c, 0.3, 0.5, 0.4), (d, 0.7, 0.2, 0.2), (e, 0.6, 0.1, 0.6)\};$$
$$R_1 = \{((a, b), 0.3, 0.2, 0.7), ((b, c), 0.3, 0.2, 0.6), ((a, d), 0.2, 0.1, 0.8), ((d, e), 0.5, 0.1, 0.6), ((b, e), 0.5, 0.1, 0.6), ((c, e), 0.2, 0.3, 0.5)\};$$
$$S_2 = \{(a, 0.7, 0.2, 0.1), (b, 0.4, 0.2, 0.6), (c, 0.8, 0.1, 0.2), (d, 0.2, 0.5, 0.4), (f, 0.6, 0.1, 0.7)\};$$
$$R_2 = \{((a, b), 0.4, 0.2, 0.6), ((b, c), 0.3, 0.3, 0.6), ((b, d), 0.2, 0.3, 0.6), ((b, f), 0.4, 0.1, 0.7), ((c, f), 0.5, 0.2, 0.7)\}.$$

Then, the graphs G_1, G_2 and their union $G_1 \cup G_2$ are being graphically presented in Fig. 5.

Example 5 (Join of T-spherical Fuzzy Graphs) Suppose $G_1' = (V_1, E_1)$ and $G_2'' = (V_2, E_2)$ are two graphs such that $V_1 = \{a, b\}$, $V_2 = \{c, d, e\}$, $E_1 = \{(c, d), (d, e)\}$ and $E_2 = \{(c, d)\}$. Let $G_1 = (S_1, R_1)$ and $G_2 = (S_2, R_2)$ be two T-spherical fuzzy graphs in reference with G_1' & G_2'', respectively, where

$$S_1 = \{(a, 0.6, 0.2, 0.3), (b, 0.5, 0.1, 0.7)\},$$
$$R_1 = \{((a, b), 0.5, 0.2, 0.7)\};$$
$$S_2 = \{(c, 0.7, 0.2, 0.5), (b, 0.5, 0.2, 0.8), (c, 0.6, 0.1, 0.6)\};$$
$$R_2 = \{((c, d), 0.5, 0.2, 0.8), ((d, e), 0.5, 0.2, 0.7)\}.$$

Then, the graphs G_1, G_2 and their join $G_1 + G_2$ are being graphically presented in Fig. 6.

– Complement of T-spherical Fuzzy Graph
The complement of a T-spherical fuzzy graph G, denoted by $\overline{G}$, is defined as $\overline{G} = (\overline{S}, \overline{R})$, where

- $\overline{V} = V$.
- $\mu_{\overline{S}}(\alpha) = \mu_S(\alpha)$
$\eta_{\overline{S}}(\alpha) = \eta_S(\alpha)$
$\nu_{\overline{S}}(\alpha) = \nu_S(\alpha);\ \forall \alpha \in V.$

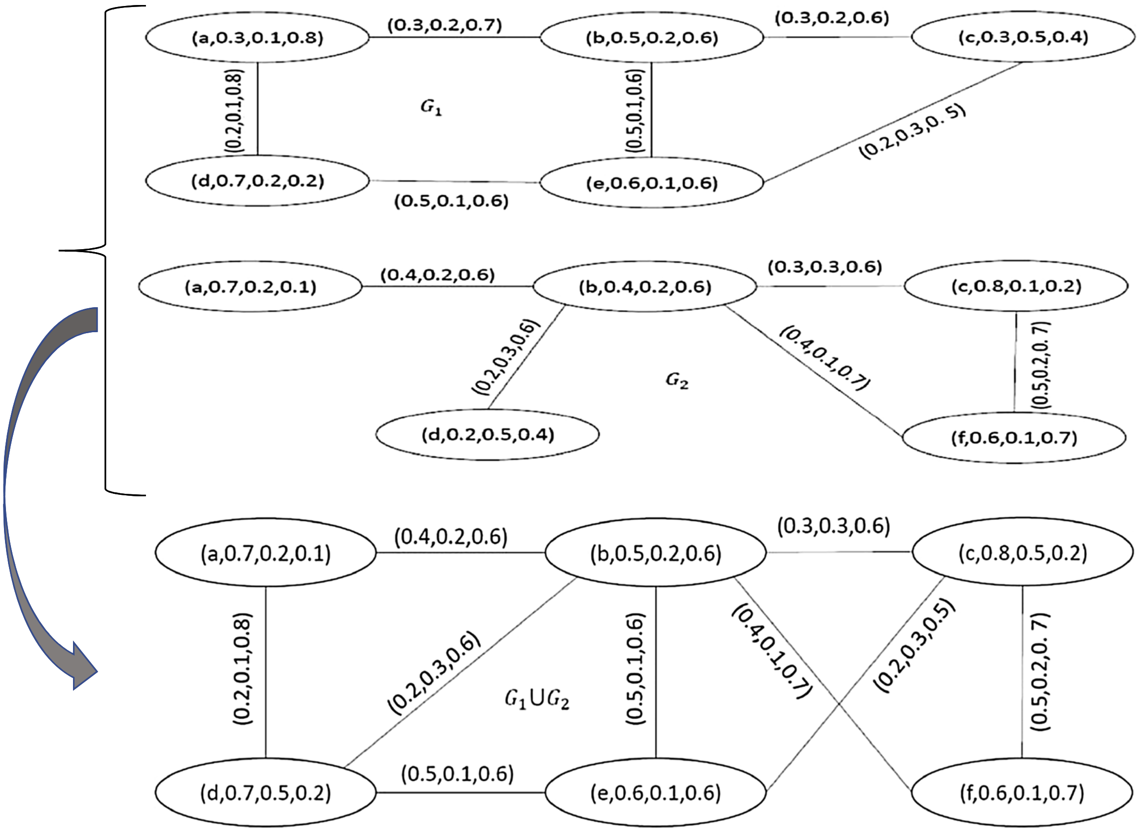

Fig. 5 Graphs G_1, G_2 and $G_1 \cup G_2$

•

$$\mu_R(\alpha,\beta) = \begin{cases} \min\left(\mu_S(\alpha), \nu_S(\beta)\right) & \text{if } \mu_R(\alpha,\beta)=0 \\ \left|\min\left(\mu_S(\alpha), \nu_S(\beta)\right) - \mu_R(\alpha,\beta)\right| & \text{if } \mu_R(\alpha,\beta)\geq 0,\ \forall(\alpha,\beta)\in E \end{cases}$$

$$\eta_R(\alpha,\beta) = \begin{cases} \min\left(\eta_S(\alpha), \eta_S(\beta)\right) & \text{if } \eta_R(\alpha,\beta)=0 \\ \left|\min\left(\eta_S(\alpha), \eta_S(\beta)\right) - \eta_R(\alpha,\beta)\right| & \text{if } \eta_R(\alpha,\beta)\geq 0,\ \forall(\alpha,\beta)\in E \end{cases}$$

$$\nu_R(\alpha,\beta) = \begin{cases} \min\left(\nu_S(\alpha), \nu_S(\beta)\right) & \text{if } \nu_R(\alpha,\beta)=0 \\ \left|\min\left(\nu_S(\alpha), \nu_S(\beta)\right) - \nu_R(\alpha,\beta)\right| & \text{if } \nu_R(\alpha,\beta)\geq 0,\ \forall(\alpha,\beta)\in E. \end{cases}$$

Example 6 Suppose $G' = (V, E)$ is a graph such that $V = \{a, b, c, d\}$ and $E = \{(a, d), (b, c), (c, d)\}$. Let $G = (S, R)$ be the corresponding T-spherical fuzzy graph of G', where

$$S = \{(a, 0.7, 0.1, 0.5), (b, 0.3, 0.2, 0.6), (c, 0.8, 0.2, 0.2), (d, 0.5, 0.2, 0.4)\};$$
$$R = \{((a, d), 0.5, 0.3, 0.4), (b, c), 0.3, 0.2, 0.6), (c, d), 0.4, 0.2, 0.4)\}.$$

Then, the graph G and its complement $\overline{G}$ are being graphically presented in Fig. 7.

5 Application of *T*-spherical Fuzzy Graphs in Decision-Making Processes

Decision making is a daily life process in which the ultimate goal is to choose the best alternative/object from the available set of alternatives/objects. In order to reach to a conclusion in real- world problems, the decision makers usually depend on various interrelating factors along with their intuition or prior expertise. The preference relation is found to be one of the most useful techniques to obtain the ranking of the alternatives in which the decision makers provide their preference over the given alternatives/criteria. For establishing the preference relation, the experts compare each pair of the alternatives from a given set of alternatives. If the information presented in the preference relation is in the form of T-spherical fuzzy numbers (TSFNs), then we introduce the concept of T-spherical fuzzy preference relation ($TSFPR$) as follows:

Definition 12 A *T-spherical fuzzy preference relation* (TSFPR) on the universal set $U = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m\}$ is given by the matrix $R = (\widetilde{r_{ij}})_{m\times m}$, where $\widetilde{r_{ij}} = \left((\alpha_i, \alpha_j), \mu(\alpha_i, \alpha_j), \eta(\alpha_i, \alpha_j), \nu(\alpha_i, \alpha_j)\right)$ for all $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, m$. For convenience, let $\widetilde{r_{ij}} = (\mu_{ij}, \eta_{ij}, \nu_{ij})$, where μ_{ij} represents the degree to

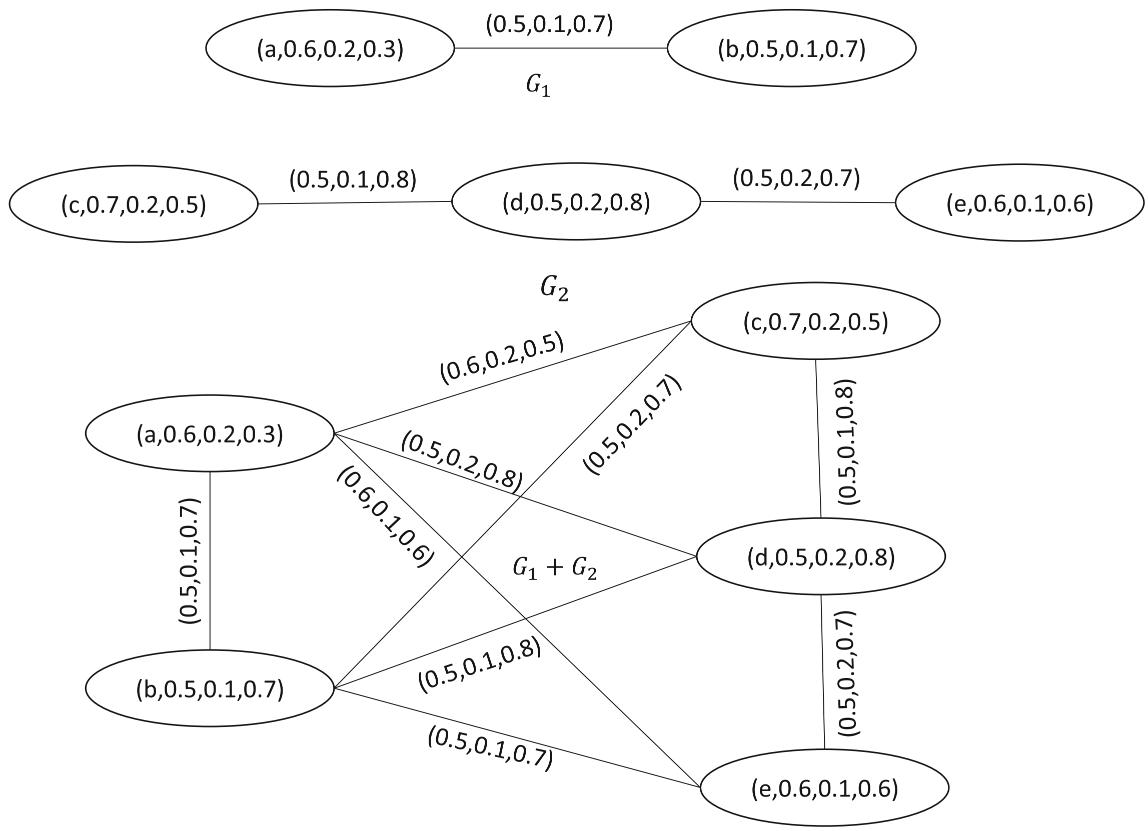

Fig. 6 Graphs G_1, G_2 and their join $G_1 + G_2$

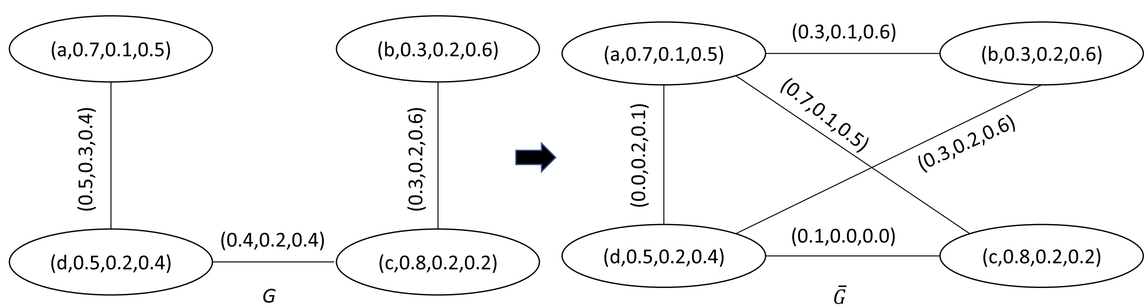

Fig. 7 Graph G and its complement graph $\overline{G}$

which the object α_i is preferred over the object α_j, η_{ij} represents the degree to which the decision maker is in confusion whether to prefer the object α_i or α_j and ν_{ij} represents the degree to which the object α_i is not preferred to the object α_j and

$$r_{ij} = \sqrt[n]{1 - (\mu_{ij}^n(\alpha) + \eta_{ij}^n(\alpha) + \nu_{ij}^n(\alpha))}$$

representing the degree of refusal, with the conditions:

$$0 \leq \mu_{ij}^n(\alpha) + \eta_{ij}^n(\alpha) + \nu_{ij}^n(\alpha) \leq 1, \quad \mu_{ij} = \nu_{ji}, \ \eta_{ij} = \eta_{ji}, \ \nu_{ij} = \mu_{ji} \ \text{and} \ \ \mu_{ii} = 1, \eta_{ii} = \nu_{ii} = 0; \quad i, j = 1, 2, \ldots, m.$$

Next, we consider the problem of partner selection in supply chain management and evaluation of service centers for the illustration of the proposed work. For the sake of simplicity and feasibility in the computation processes, we take the value of n to be 2.

5.1 Selection of Partner in Supply Chain Management

In a supplier selection problem, values of the attributes/criteria are not supposed to be independent, i.e., the interrelationship between attribute values should also be taken into account while processing the information. For this purpose, incorporating the proposed notion of spherical fuzzy graph could be worth enough to express the interrelationships among the attributes/criteria. We need to focus on the role of the critical factors involved in assessing the available potential partners for a company. This eventually depends on the strategic relationships among the companies concerning with the supply chain. Because of the synchronized coordination, the companies get benefits from the lower cost, the lower inventory, the information sharing and thus forming a sharp competitive edge. Suppose that the four critical factors (F_i ; $i = 1, 2, 3, 4$) involved for the desired assessment are as [36]:

- F_1: "response time and supply capacity";
- F_2: "quality and technical skills";
- F_3: "price and cost";
- F_4: "service level."

In order to rank the above four factors, F_i ($i = 1, 2, 3, 4$), a committee of three equipotential experts $\{e_1, e_2, e_3\}$ is constituted whose weight vector $w = (1/3, 1/3, 1/3)$. Here, in view of the proposed T-spherical fuzzy graphs with preference relation, we present an algorithm for solving the above-stated decision-making problem whose flowchart is given in Fig. 8.

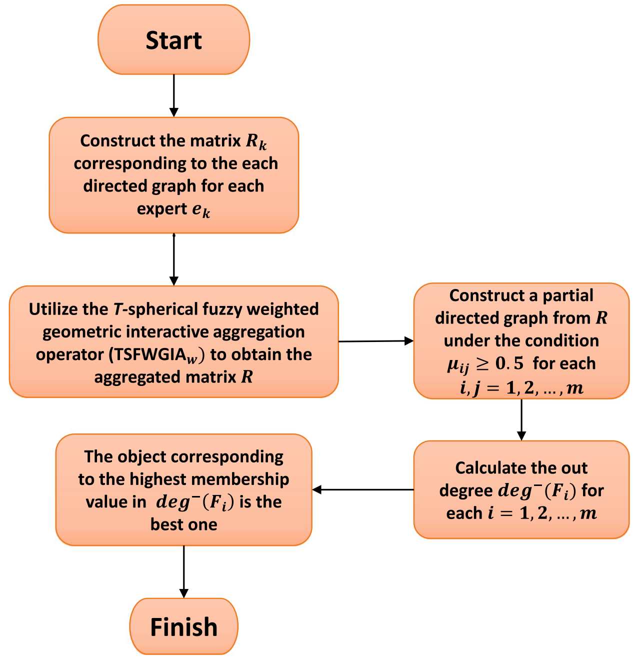

Fig. 8 Algorithm for assessment of critical factors in supply chain

Procedural Steps of the Proposed Algorithm:

– **Step 1:** The decision makers compare these involved factors among themselves and provide the initial information for the computation in the form of T-spherical fuzzy preference relations $R_k = (\widetilde{r_{ij}}^{(k)})_{4\times 4}$ ($k = 1, 2, 3$), which are given by:

$$R_1 = \begin{pmatrix} (1.0, 0.0, 0.0) & (0.5, 0.2, 0.6) & (0.8, 0.1, 0.3) & (0.6, 0.2, 0.4) \\ (0.6, 0.2, 0.5) & (1.0, 0.0, 0.0) & (0.5, 0.3, 0.3) & (0.3, 0.2, 0.7) \\ (0.3, 0.1, 0.8) & (0.3, 0.3, 0.5) & (1.0, 0.0, 0.0) & (0.8, 0.2, 0.4) \\ (0.4, 0.2, 0.6) & (0.7, 0.2, 0.3) & (0.4, 0.2, 0.8) & (1.0, 0.0, 0.0) \end{pmatrix};$$

$$R_2 = \begin{pmatrix} (1.0, 0.0, 0.0) & (0.8, 0.2, 0.3) & (0.9, 0.1, 0.2) & (0.1, 0.2, 0.9) \\ (0.3, 0.2, 0.8) & (1.0, 0.0, 0.0) & (0.7, 0.2, 0.2) & (0.8, 0.4, 0.2) \\ (0.2, 0.1, 0.9) & (0.2, 0.2, 0.7) & (1.0, 0.0, 0.0) & (0.9, 0.1, 0.1) \\ (0.9, 0.2, 0.1) & (0.2, 0.4, 0.8) & (0.1, 0.1, 0.9) & (1.0, 0.0, 0.0) \end{pmatrix};$$

and

$$R_3 = \begin{pmatrix} (1.0, 0.0, 0.0) & (0.7, 0.2, 0.3) & (0.5, 0.3, 0.6) & (0.8, 0.1, 0.2) \\ (0.3, 0.2, 0.7) & (1.0, 0.0, 0.0) & (0.4, 0.2, 0.6) & (0.5, 0.2, 0.5) \\ (0.6, 0.3, 0.5) & (0.6, 0.2, 0.4) & (1.0, 0.0, 0.0) & (0.2, 0.1, 0.9) \\ (0.2, 0.1, 0.8) & (0.5, 0.2, 0.5) & (0.9, 0.1, 0.2) & (1.0, 0.0, 0.0) \end{pmatrix}.$$

– **Step 2:** Here, we use the recent aggregation operator, termed as T-spherical fuzzy weighted geometric interactive aggregation operator, given by Garg et al. [37]:

$$T - SFWGIA_w(\widetilde{r_{ij}}^{(1)}, \widetilde{r_{ij}}^{(2)}, \ldots, \widetilde{r_{ij}}^{(m)}) = \left(\sqrt[n]{\prod_{j=1}^{m}(1-\mu_j^n)^{w_j} - \prod_{j=1}^{m}(1-\mu_j^n-\eta_j^n-\nu_j^n)^{w_j} - \prod_{j=1}^{m}(\eta_j^n)^{w_j}}, \sqrt[n]{1-\prod_{j=1}^{m}(1-\eta_j^n)^{w_j}}, \sqrt[n]{1-\prod_{j=1}^{m}(1-\nu_j^n)^{w_j}} \right).$$

Next, using the aggregation operator cited in Step 2, we aggregate the three T-spherical fuzzy preference relations R_1, R_2 and R_3 into a single preference relation, which is computed as follows:

$$R = \begin{pmatrix} (1.0, 0.0, 0.0) & (0.68, 0.20, 0.44) & (0.76, 0.20, 0.42) & (0.49, 0.17, 0.68) \\ (0.40, 0.20, 0.69) & (1.0, 0.0, 0.0) & (0.54, 0.24, 0.42) & (0.59, 0.29, 0.53) \\ (0.36, 0.19, 0.79) & (0.40, 0.24, 0.56) & (1.0, 0.0, 0.0) & (0.77, 0.17, 0.38) \\ (0.56, 0.17, 0.62) & (0.51, 0.29, 0.61) & (0.48, 0.14, 0.77) & (1.0, 0.0, 0.0) \end{pmatrix}.$$

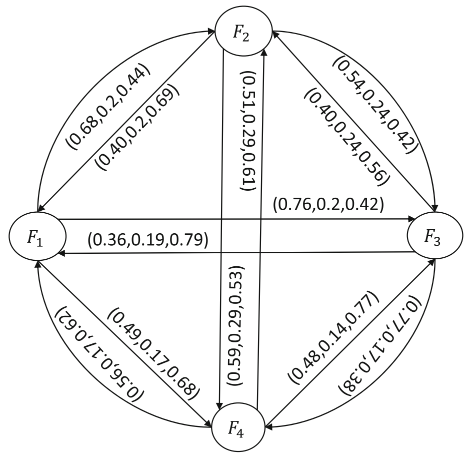

Fig. 9 Directed graph of TSFPR R

Now, we draw a directed graph corresponding to the above-obtained preference relation R given by Fig. 9.

– **Step 3:** In this step, we construct another partial directed graph under the additional condition of $\mu_{ij} \geq 0.5$, $(i, j = 1, 2, 3, 4)$, given by Fig. 10.
– **Step 4:** Calculate the out-degrees $\deg^{-}(F_i)$, $(i = 1, 2, 3, 4)$ of all the factors for the partial directed graph obtained in Step 3 as follows:

$$\deg^{-}(F_1) = (1.44, 0.40, 0.86),$$
$$\deg^{-}(F_2) = (1.13, 0.53, 0.96),$$
$$\deg^{-}(F_3) = (0.77, 0.17, 0.38),$$
$$\deg^{-}(F_4) = (1.07, 0.46, 1.23).$$

– **Step 5:** Finally, on the basis of the highest membership degrees obtained in Step 4, we get the ranking of the $F_i's$ as follows:

$$F_1 > F_2 > F_4 > F_3.$$

Thus, we conclude that the critical factor F_1 ("response time and supply capacity") is the most influential factor.

In contrast, if we consider a similar problem of supply chain management [34] to assess the potential partner of the firm, where the information has been taken in the form of Pythagorean fuzzy numbers and Pythagorean fuzzy graphs along with the preference relation and aggregation operator, we observe that the results obtained are as follows:

$$\text{out}-d(F_1) = (1.35, 1.1),\ \ \text{out}-d(F_2) = (1.27, 1.05),$$
$$\text{out}-d(F_3) = (0.0, 0.0),\ \ \text{out}-d(F_4) = (1.69, 1.76).$$

This shows that the critical factor F_4 ("service level") is the most influential factor.

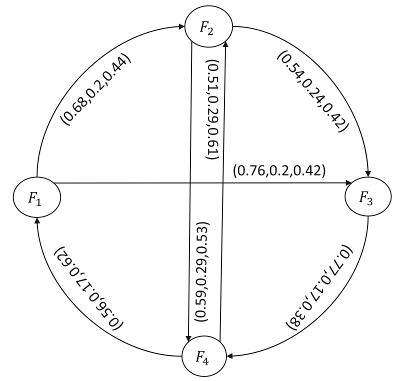

Fig. 10 Partial directed graph of fused TSFPR R

Limitations of the Existing Methodologies

- The idea of Pythagorean fuzzy graph is not fully capable enough to depict the human opinion completely as the full information specification may contain the component of refusal in various problems and applications. In such cases, the decision makers are bound to give their opinion under a constraint.
- The problem which we have considered above contains the information in the form of all the four components of fuzziness which cannot be dealt by intuitionistic fuzzy graph or Pythagorean fuzzy graph as well. This may be viewed in Table 1.
- Also, the existing techniques related to Pythagorean fuzzy graphs and operations lack to consider the related dependability in the incomplete information which has a degree of refusal.

Comparative Remarks

The following are the major points based on which the comparison between the existing techniques and the proposed technique can be understood:

- The difference in the results obtained above shows that the additional component of the information has been duly taken care in the spherical fuzzy information. Thus, we see that it is not possible to solve the problem under consideration where the information is in the form of T-spherical fuzzy number (due to the presence of the addition component of fuzziness, that is, degree of refusal) by using the Pythagorean fuzzy graphs [34].
- The MCDM problems discussed in [14] and [15] do not assimilate all the information parameters and do not use any aggregation operator.
- Hence, the combination of the proposed T-spherical fuzzy graph and interactive aggregation operator [37] used for the partner selection problem can handle the broader space of information and has wider applicability

Table 1 Comparison with other existing approaches

Approaches	μ	ν	η	r	Aggreagtion operator	Information covered
FG[11]	✓	✓	×	×	×	×
IFG[23]	✓	✓	✓	×	×	Partial
PyFG[21]	✓	✓	✓	×	✓	Partial
TSFG (proposed)	✓	✓	✓	✓	✓	Complete

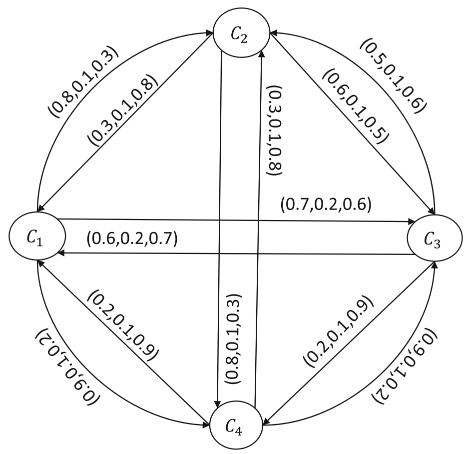

Fig. 11 Directed graph network of T-spherical fuzzy preference relation

in the real-world situations. This can further be analyzed in view of Table 1.

5.2 Evaluation of Service Centers

Next, we consider another decision-making problem where the evaluation of the four service centers $C_i,\ i = 1, 2, 3, 4$ of a city is to be done under a uniformity in the choice [weight vector $w = (1/4, 1/4, 1/4, 1/4)$]. The decision makers compare the results on the basis of the criterion "quality of service" and provide the judgment by using the directed graph network in Fig. 11 whose vertices represent the service centers.

Based on the T-spherical fuzzy graph, preference relation, aggregation operator and score function, we propose an algorithm for the assessment of the service centers on the basis of "quality of service" provided by them. The flowchart of the algorithm is presented in Fig. 12.

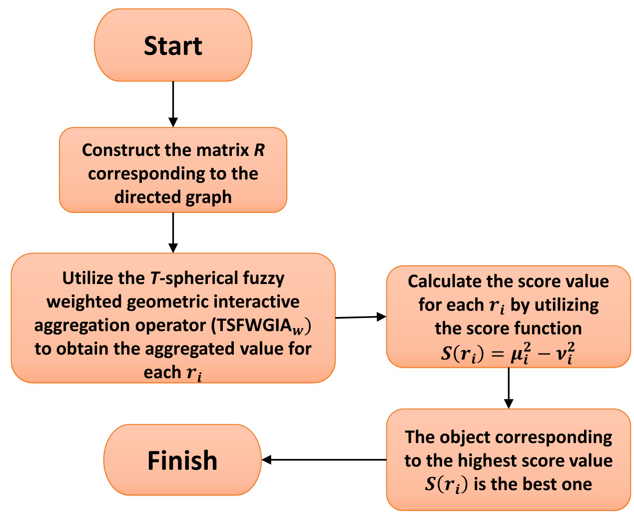

Fig. 12 Algorithm for solving the evaluation problem

Procedural Steps of the Proposed Algorithm:

– **Step 1:** Based on the decision maker's opinion, construct the preference relation matrix R as follows:

– **Step 2:** We use the recent aggregation operator, termed as T-spherical fuzzy weighted geometric interactive aggregation operator, given by Garg et al. [37]:

$$\tilde{r_i} = T - \mathrm{SFWGIA}_w(\widetilde{r_{i1}}, \widetilde{r_{i2}}, \ldots, \widetilde{r_{im}}) = \Bigg(\sqrt[n]{\prod_{j=1}^{m}(1-\mu_j^n)^{w_j} - \prod_{j=1}^{m}(1-\mu_j^n-\eta_j^n-\nu_j^n)^{w_j} - \prod_{j=1}^{m}(\eta_j^n)^{w_j}}\ \sqrt[n]{1-\prod_{j=1}^{m}(1-\eta_j^n)^{w_j}},\ \sqrt[n]{1-\prod_{j=1}^{m}(1-\nu_j^n)^{w_j}} \Bigg),$$

for $i = 1, 2, \ldots, m$. Aggregate all the $r_{ij},\ j = 1, 2, 3, 4$ corresponding to the service center (C_i) [weight vector $w = (1/4, 1/4, 1/4, 1/4)$], and then we get T-spherical fuzzy number r_i corresponding to the service center (C_i) with preference over all the other service centers:

$$r_1 = (0.91, 0.14, 0.42),\ \ r_2 = (0.79, 0.10, 0.61),$$
$$r_3 = (0.63, 0.14, 0.78),\ \ r_4 = (0.64, 0.10, 0.77).$$

$$R = (r_{ij})_{4\times4} = \begin{pmatrix} (1.0, 0.0, 0.0) & (0.8, 0.1, 0.3) & (0.7, 0.2, 0.6) & (0.9, 0.1, 0.2) \\ (0.3, 0.1, 0.8) & (1.0, 0.0, 0.0) & (0.6, 0.1, 0.5) & (0.8, 0.1, 0.3) \\ (0.6, 0.2, 0.7) & (0.5, 0.1, 0.6) & (1.0, 0.0, 0.0) & (0.2, 0.1, 0.9) \\ (0.2, 0.1, 0.9) & (0.3, 0.1, 0.8) & (0.9, 0.1, 0.2) & (1.0, 0.0, 0.0) \end{pmatrix}.$$

– **Step 3:** In order to calculate the score value for each r_i, $i = 1, 2, 3, 4$, we use the score function [11] in reference with spherical fuzzy set given by $S(r_i) = \mu_i^2 - \nu_i^2$:

$$S(r_1) = 0.7645, \ S(r_2) = 0.252, \ S(r_3) = -0.2115,$$
$$S(r_4) = -0.1833.$$

– **Step 4:** Finally, based on the score values computed in Step 3, we rank the service centers as follows:

$$C_1 > C_2 > C_4 > C_3.$$

Hence, we see that the service center C_1 is the best one on the basis of "quality of service."

The limitations of the existing techniques in reference with evaluation problem (decision making) are the same as we had in the partner selection problem (group decision making).

Comparative Remarks

The following are the major points based on which the comparison between the existing techniques and the proposed technique can be understood:

- In contrast with the selection process problem related to the evaluation of the hospitals discussed by Naz et al. [34], it has been observed that the information in this case has been taken as Pythagorean fuzzy numbers and Pythagorean fuzzy graphs. In the literature, one may find similar problems of selection processes which has been studied by various researchers using the intuitionistic fuzzy number/graph.
- Certainly, these are not capable to capture the information where there is an additional involvement of "degree of refusal". The proposed notion of T-spherical fuzzy graph and interactive aggregation operator [37] can better handle such decision-making processes which has the broader span and wider applicability.

Therefore, the discussions over implementing the T-spherical fuzzy graphs in supply chain management problem and evaluation problem clearly show that the proposed work handled the generalized framework in an effective and consistent way.

6 Comparison and Advantages of the Proposed T-spherical Fuzzy Graph

Based on the above propositions, calculations and applications, the following are the important comparative remarks and advantages of utilizing the notion of T-spherical fuzzy graphs and their operations:

- The notions of fuzzy sets and intuitionistic fuzzy sets have their own limitations that they are not capable to capture the full information specification in various situation. The condition of membership degree and non-membership degree, i.e., $0 \leq \mu_I(x) + \nu_I(x) \leq 1$, may not be satisfied in some cases where the decision makers are not bound to give their opinion under a constraint.
- In order to overcome this drawback and to capture the information in a wider sense, Yager [3] proposed the concept of Pythagorean fuzzy set as an extended version where $0 \leq \mu_M^2(x) + \nu_M^2(x) \leq 1$. Various applications of the concept of Pythagorean fuzzy are available in the literature.
- As discussed in the voting system, there is an additional term called "degree of refusal" involved in an uncertain information which can be handled by the picture fuzzy set [10], i.e., $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$. We may refer to the definitions given in Sect. 2 for a better understanding in this regard. This proposition has its own limitation in reference with the flexibility own by a decision maker.
- For instance, if a decision maker provides the evaluation information with membership degree as 0.6, neutral degree as 0.3 and the non-membership degree as 0.5, then it may be observed that the picture fuzzy number fails to handle such information because $0.6 + 0.3 + 0.5 > 1$. However, in view of the definition of spherical fuzzy number, $0.6^2 + 0.3^2 + 0.5^2 < 1$, i.e., the spherical fuzzy number is capable enough to represent such information, as being observed in Table 2.
- In this way, the experts/decision makers may allocate the membership values of their own choice. This makes the decision makers more enable for providing their input best suit to their domain of reference.
- The selection processes studied in Sect. 5 have well utilized the proposed notion of T-spherical fuzzy graphs and the proposed algorithms in order to provide a generalization feature/framework to make a strong impact on the applications.
- Therefore, the proposed graphs and operations have capabilities to address the related dependability on the imprecise information which has a degree of refusal in a more reliable and superior manner. Table 1 clearly explains the advantages of the proposed approach in contrast with the existing approaches.
- Various other problems related to stocks investment analysis, service quality of airlines, the authority selection in investment banking, electronic learning factor's evaluation and others may be well studied and discussed using the proposed T-spherical fuzzy graphs and the methodologies.

Table 2 Need to address the problem arises in IFSs, PyFSs and PFSs

R	C_1	C_2	C_3	C_4
C_1	$(1.0 + 0.0 + 0.0 = 1)$	$(0.40 + 0.20 + 0.69 > 1)$	$(0.36 + 0.19 + 0.79 > 1)$	$(0.56 + 0.17 + 0.62 > 1)$
C_2	$(0.68 + 0.20 + 0.44 > 1)$	$(1.0 + 0.0 + 0.0 > 1)$	$(0.40 + 0.24 + 0.56 > 1)$	$(0.51 + 0.29 + 0.61 > 1)$
C_3	$(0.76 + 0.20 + 0.42 > 1)$	$(0.54 + 0.24 + 0.42 > 1)$	$(1.0 + 0.0 + 0.0 > 1)$	$(0.48 + 0.14 + 0.77 > 1)$
C_4	$(0.49 + 0.17 + 0.68 > 1)$	$(0.59 + 0.29 + 0.53 > 1)$	$(0.77 + 0.17 + 0.38 > 1)$	$(1.0 + 0.0 + 0.0 > 1)$

7 Conclusions and Scope for Future Work

A Pythagorean fuzzy graph network model is capable to describe the problems with uncertainty, imprecision and inconsistent information in contrast with the classical fuzzy/intuitionistic fuzzy models. However, the flexibility which arises because of degree of refusal has not been taken care by Pythagorean fuzzy graphs. Here, we have proposed a new kind of graph called T-spherical fuzzy graphs and their operations (e.g., Cartesian product, composition, union, join and complement) which are found to be worthy enough. This has the feature to model and handle the component of degree of refusal which provides a wider coverage and wider geometrical span. The operations have also been defined and well explained with suitable graph-theoretic examples. In view of T-spherical fuzzy preference relation, two algorithms have been proposed for solving the problems of supply chain management and evaluation process. Implementation of the proposed algorithms has been illustrated through numerical examples.

The concept of energy of the T-spherical fuzzy graph and various other graph-theoretic features, e.g., isomorphism, planarity, duality, adjacency matrix, regularity, hypergraphs, etc., may further be extended and applied in the field of designing an engineering system, system analysis, etc. Also, an extension to hesitant T-spherical fuzzy graph and T-spherical fuzzy soft graph may also be explored with suitable applications.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors

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