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Eigen spherical fuzzy set and its application to decision-making problem

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Abstract. An eigen fuzzy set of a fuzzy relation is often invariant under different computational aspects. The present research introduces a novel concept of eigen spherical fuzzy set of spherical fuzzy relations along with various composition operators for the first time. This study proposed two algorithms to determine the greatest eigen spherical fuzzy sets and least eigen spherical fuzzy sets using the $max - min$ and $min - max$ composition operators, respectively, and illustrated the steps through flow charts. Further, two numerical examples related to different fields of decision-making problems were taken into account for illustrating the proposed methodology. The scope of future work in the field of image information retrieval, genetic algorithm for image reconstruction, and notion of eigen spherical fuzzy soft sets/matrices was duly outlined. The comparative remarks and advantages of the proposed eigen spherical fuzzy sets were also included for better readability.

1. Introduction

Researchers in the field of fuzzy sets and information are well aware that various generalizations of the notions of fuzzy sets [1] and Intuitionistic Fuzzy Sets (IFS) [2] play roles in modeling the uncertainties and hesitancy inherent in many practical circumstances with a broader range of various applications, particularly in the study of patterns and information systems. Essentially, such generalizations provide a formal approach to dealing with real-life problems in which the source of impreciseness is the lack of sharply defined criteria of class membership instead of having random variables. Yager [3] revealed that the existing structures of fuzzy set and intuitionistic fuzzy set were not capable enough to depict the human opinion in a more practical/broader sense and, therefore, they introduced the notion of Pythagorean Fuzzy Sets (PyFSs) which effectively enlarged the span of information by introducing the new conditional constraint. Various other operations over inter-valued PyFSs were given by Peng [4]. The concepts of membership/belongingness (yes), non-membership/non-belongingness (no), and indeterminacy/neutral (abstention) have been well described by the definition of intuitionistic fuzzy sets as well as by the PyFSs. Consider an example of a voting system where voters can be categorized into four different classes: one who votes for (yes), one who votes against (no), one who neither votes for nor against (abstention), and one who refuses to vote (refusal). It may be noted that the concept of 'refusal' is not being taken into account by any of the sets stated

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above. In order to deal with such circumstances and develop a concept that would be sufficiently close to human's nature of flexibility, Cuong [5,6] introduced the concept of Picture Fuzzy Set (PFS) in which all the four parameters, i.e., degree of membership, degree of indeterminacy (neutral), degree of nonmembership, and the degree of refusal were taken into account.

Recently, Mahmood et al. [7] introduced the notion of Spherical Fuzzy Set (SFS) and T-Spherical Fuzzy Set (TSFS) that gave additional strength to the idea of PFSs by broadening/enlarging the space for the grades of all the four parameters. Next, Kifayat et al. [8] studied the geometrical comparison of fuzzy sets, intuitionistic fuzzy sets, PyFSs, and picture fuzzy sets with spherical and TSFS. Also, they studied various existing similarity measures for intuitionistic fuzzy sets and found that PFSs had some limitations and could not be applied to the broader setup of the spherical fuzzy environment. Further, they proposed various types of similarity measures for TSFS with their useful applications in various fields. The evolution process of the generalizations and extensions of a fuzzy set is summarized in Figure 1.

In the field of pure and applied sciences, the mathematical notion of relation plays a key role in establishing the connections between objects, states, and events. Fuzzy relations are the generalizations of the concept of binary relations. The notion of fuzzy relation was first introduced by Zadeh [9] with fuzzy equivalence (similarity) relation and provided the concept of fuzzy ordering along with some basic properties.

Sanchez [10,11] described the role of invariant fuzzy sets linked with a given fuzzy relation using the composition of fuzzy relations and introduced the notion of eigen fuzzy set of a fuzzy relation. Further, Sanchez used *max* - *min* composition to determine the Greatest Eigen Fuzzy Set (GEFS) associated with fuzzy relation by providing three major algorithms. Several practical and successful applications of eigen fuzzy sets in the field of image analysis (image reconstruction, image information retrieval, image decomposition) [12–14], genetic algorithm [15], medicine (drug effectiveness levels) [16], fuzzy Markov chain and decision-making [17,18], etc.

To span the flexibility of human opinions with revised conditional constraints, we propose an extension as a new paradigm called Eigen Spherical Fuzzy Sets (ESFS) of spherical fuzzy relations. For application purposes, two new methods for determining the Greatest Eigen Spherical Fuzzy Sets (GESFS) and Least Eigen Spherical Fuzzy Sets (LESFS) by using the $max - min$ and $min - max$ composition have been proposed.

The rest of the paper is organized as follows: A brief literature review related to the eigen fuzzy sets

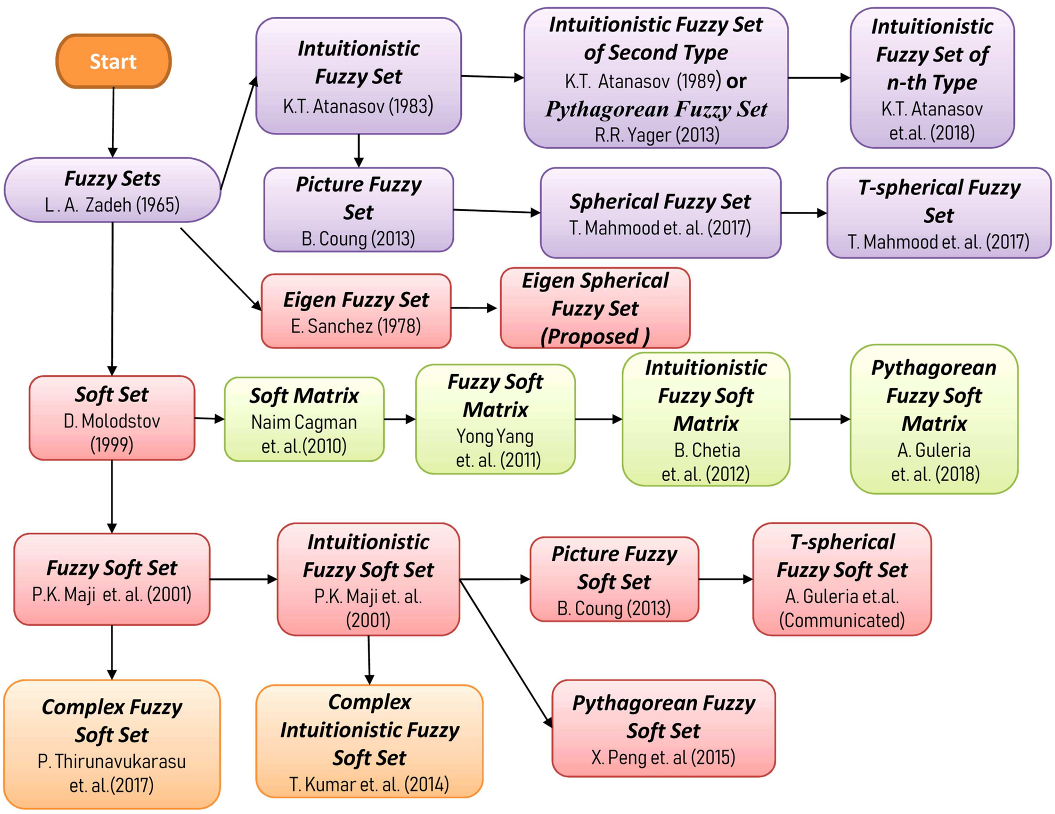

Figure 1. Extensions and generalizations of fuzzy set.

and their applications is presented in Section 2. Some fundamental background and definitions are studied in Section 3. The notion of spherical fuzzy relations and functioning of their composition operators are introduced in Section 4. In Section 5 shows the proposed novel concept of ESFS of spherical fuzzy relation along with the algorithms to determine the GESFS and LESFS using the well-defined composition operators. This has been well illustrated using examples. Section 6 presents the implementation of the proposed algorithms by solving two examples related to decision-making. In Section 7, some potential directions and guidelines for future works in different application fields are provided in brief. The comparative remarks and the advantages of the proposed ESFS are discussed in Section 8. Finally, the paper is concluded in Section 9.

2. Literature review

In literature, various researchers have studied fuzzy relational calculus as an application of fuzzy relation to obtain possible solutions to fuzzy relation equations [19–21] and proposed the notion of eigen fuzzy set being invariant in the associated fuzzy relation. Consequently, the problem of finding the GEFS associated with a relation has been dealt. This has generated a considerable amount of interest for researchers for further investigations. Goetschel and Voxman [22] extended the concept and results of eigen fuzzy set with the eigen fuzzy numbers by making slight modifications to the definition of fuzzy number given by Dubois and Prade [23]. Fernandez et al. [24] generalized the results of Sanchez [19] to determine the greatest T-eigen fuzzy set of fuzzy relations and studied some algebraic properties.

The applications of eigen fuzzy sets have successfully been carried out by researchers in the second half of the 1970s which boosted the reader's interest significantly in this area. Amagaza and Tazaki and Amagasa [25] studied heuristic structure synthesis using eigen fuzzy sets. Cao [26] presented an algorithm for finding eigen fuzzy sets of a fuzzy matrix. The concept of general finite state fuzzy Markov chains in connection with the GEFS of the transition matrix was introduced by Avrachenkov and Sanchez [17] and was finally linked to fuzzy Markov decision-making processes.

Nobuhara and Hirota [14] studied the GEFS of $max - min$ composition and an adjoint concept of GEFS, i.e., the smallest eigen fuzzy set of adjoint $max - min$ composition of a fuzzy relation using the principal component analysis of images. Di Martino et al. [13] introduced the Least Eigen Fuzzy Set (LEFS) based on the $min - max$ composition and established that both GEFS and LEFS were useful in image information retrieval. Further, they compared the original image with retrieved images by introducing a similarity measure based on GEFS and LEFS. Next, Nobuhara et al. [12] proposed two algorithms for the image reconstruction process based on the convex combination of eigen fuzzy sets of $max - min$ and $min - max$ compositions and using the eigen fuzzy sets generated by a permutation matrix, where the images are treated as fuzzy relations. Based on the eigen fuzzy set of fuzzy relations, Di Martino and Sessa [15] proposed a genetic algorithm for image reconstruction in which GEFS and LEFS were used to calculate the highest value of the fitness.

In pharmaceutical applications, the evaluation of medicine action levels has been studied using the eigen FS [27]. Further, establishing fuzzy relations between the possible symptoms, Andersson [16] utilized the greatest and LEFS to measure the drug effectiveness levels.

3. Preliminaries

In this section, we recall and present some fundamental concepts in connection with SFS and fuzzy relations, which are well known in literature. The following notions explain the generalization process from intuitionistic fuzzy sets to SFS:

Let U be the universe of discourse with $\mu_A : U \to [0,1]$ and $\nu_A : U \to [0,1]$ being the degrees of membership and non-membership, respectively. The set $A = \{< x, \mu_A(x), \nu_A(x) >| \, x \in U\}$ is called:

- ***Intuitionistic fuzzy set* [2].** A in U if it satisfies the condition $0 \leq \mu_A(x)+\nu_A(x) \leq 1$ with the degree of indeterminacy given by:

$$\pi_I(x) = 1 - \mu_I(x) - \nu_I(x).$$

- ***Pythagorean Fuzzy Set (PyFS)* [3] or *intuitionistic fuzzy set of the second type* [28].** A in U if it satisfies the condition $0 \leq \mu_A^2(x)+\nu_A^2(x) \leq 1$ with the degree of indeterminacy given by $\pi_A(x) = \sqrt{1 - \mu_A^2(x) - \nu_A^2(x)}$.

In order to have further generalization, we consider the universe of discourse U with $\mu_A : U \to [0,1]$, $\eta_A : U \to [0,1]$, and $\nu_A : U \to [0,1]$ being the degree of membership, degree of neutral membership (abstention), and degree of non-membership, respectively. The set $A = \{< x, \mu_A(x), \eta_A(x), \nu_A(x) >| \, x \in U\}$ is called:

- ***Picture fuzzy set* [5].** A in U if it satisfies the condition $\mu_A(x)+\eta_A(x)+\nu_I(x) \leq 1$ with the degree of refusal given by:

$$r_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)).$$

- ***SFS* [7].** A in U if it satisfies the condition $\mu_A^2(x)+\eta_A^2(x) + \nu_A^2(x) \leq 1$ with the degree of refusal given by:

$$r_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)).$$

Throughout this paper, $SFS(U)$ denotes a set of all the SFS on U.

Definition 1 (fuzzy relation) [9]. *A fuzzy relation R on a fuzzy set X is a fuzzy subset of $X \times X$, i.e.:*

$$R = \{(x_1, x_2), \mu_R(x_1, x_2) \mid x_1, x_2 \in X\},$$

such that $0 \le \mu_R + \nu_R \le 1$ and $\mu_R, \nu_R \in [0,1]$. We denote $FR(X \times X)$ as a collection of all the fuzzy relations on X.

Definition 2 (eigen fuzzy set) [10]. *Let R be a fuzzy relation on the elements of a FS X, i.e., $R \in FR(X \times X)$. Consider $A \subseteq X$. Then, A is said to be an eigen FS associated with the relation R if $A = \{x, \mu_A(x)\}$ satisfies the condition $A \circledast R = A$ with $\mu_A(x) \in [0,1]$, where $\circledast$ is any composition operator.*

4. Spherical fuzzy relation and composition operators

In this section, we are proposing various composition operators for spherical fuzzy relations. Here, we first define the spherical fuzzy relation [7] as follows.

A spherical fuzzy relation R between two SFS X and Y is a spherical fuzzy subset of $X \times Y$, given by:

$$R = \{(x,y), \mu_R(x,y), \eta_R(x,y), \nu_R(x,y) \mid x \in X, y \in Y\},$$

such that $0 \le \mu_R^2 + \eta_R^2 + \nu_R^2 \le 1$ and $\mu_R, \eta_R, \nu_R \in [0,1]$. In this paper, $SFR(X \times Y)$ denotes the set of all the spherical fuzzy relations between X and Y.

Let $R_1 \in SFR(X \times Y)$ and $R_2 \in SFR(Y \times Z)$ be two spherical fuzzy relations. We define various composition operators for the spherical fuzzy relations R_1 and R_2 as follows:

- **Max-Min composition of spherical fuzzy relations.** The $max - min$ composition relation of R_1 and R_2, denoted by $R_1 \circ R_2 \in SFR(X \times Z)$, is defined as:

$$R_1 \circ R_2 = \{(x,z), \mu_{R_1 \circ R_2}(x,z), \eta_{R_1 \circ R_2}(x,z), \nu_{R_1 \circ R_2}(x,z) \mid x \in X, z \in Z\},$$

where:

$$\mu_{R_1 \circ R_2} = \max\{\min(\mu_{R_1}(x,y), \mu_{R_2}(y,z))\};$$

$$\eta_{R_1 \circ R_2} = \min\{\min(\eta_{R_1}(x,y), \eta_{R_2}(y,z))\};$$

$$\nu_{R_1 \circ R_2} = \min\{\max(\nu_{R_1}(x,y), \nu_{R_2}(y,z))\}.$$

- **Min-Max composition of spherical fuzzy relations:**
The $min - max$ composition relation of R_1 and R_2, denoted by $R_1 \bullet R_2 \in SFR(X \times Z)$, is defined as follows:

$$R_1 \bullet R_2 = \{(x,z), \mu_{R_1 \bullet R_2}(x,z), \eta_{R_1 \bullet R_2}(x,z), \nu_{R_1 \bullet R_2}(x,z) \mid x \in X, z \in Z\},$$

where:

$$\mu_{R_1 \bullet R_2} = \min\{\max(\mu_{R_1}(x,y), \mu_{R_2}(y,z))\};$$

$$\eta_{R_1 \bullet R_2} = \min\{\min(\eta_{R_1}(x,y), \eta_{R_2}(y,z))\};$$

$$\nu_{R_1 \bullet R_2} = \max\{\min(\nu_{R_1}(x,y), \nu_{R_2}(y,z))\}.$$

- **Max-Min average composition of spherical fuzzy relations.** The $max - min$ average composition relation of R_1 and R_2, denoted by $R_1 \Phi R_2 \in SFR(X \times Z)$, is defined as follows:

$$R_1 \Phi R_2 = \{(x,z), \mu_{R_1 \Phi R_2}(x,z), \eta_{R_1 \Phi R_2}(x,z), \nu_{R_1 \Phi R_2}(x,z) \mid x \in X, z \in Z\},$$

$$\mu_{R_1 \Phi R_2} = \max\left\{\frac{\mu_{R_1}(x,y) + \mu_{R_2}(y,z)}{2}\right\};$$

$$\eta_{R_1 \Phi R_2} = \min\left\{\frac{\eta_{R_1}(x,y) + \eta_{R_2}(y,z)}{2}\right\};$$

$$\nu_{R_1 \Phi R_2} = \min\left\{\frac{\nu_{R_1}(x,y) + \nu_{R_2}(y,z)}{2}\right\}.$$

- **Min-Max average composition of spherical fuzzy relations.** The $min - max$ average composition relation of R_1 and R_2, denoted by $R_1 \Psi R_2 \in SFR(X \times Z)$, is defined as follows:

$$R_1 \Psi R_2 = \{(x,z), \mu_{R_1 \Psi R_2}(x,z), \eta_{R_1 \Psi R_2}(x,z), \nu_{R_1 \Psi R_2}(x,z) \mid x \in X, z \in Z\},$$

$$\mu_{R_1 \Psi R_2} = \min\left\{\frac{\mu_{R_1}(x,y) + \mu_{R_2}(y,z)}{2}\right\};$$

$$\eta_{R_1 \Psi R_2} = \min\left\{\frac{\eta_{R_1}(x,y) + \eta_{R_2}(y,z)}{2}\right\};$$

$$\nu_{R_1 \Psi R_2} = \max\left\{\frac{\nu_{R_1}(x,y) + \nu_{R_2}(y,z)}{2}\right\}.$$

Remark: Klement et al. [29,30] studied some basic *triangular norm* (t-*norm*) & *triangular conorm* (t-*conorm*), their types, and various properties. Various other operators may also be defined over spherical fuzzy relations. In the present work, the combination of maximum operator (t-*conorm*) and minimum operator (t-*norm*) has been taken into account. Various other combinations using other types of t-*norm* and t-*conorm* may can also be utilized in future.

5. ESFS and algorithms for GESFS and LESFS

This section introduces the notion of ESFS and provides necessary steps of appropriate methods for finding the GESFS and the LESFS using numerical examples.

Let R be a spherical fuzzy relation between two SFS X and Y, i.e., $R \in SFR(X \times Y)$ and $S \in SFS(U)$ be SFS. The composition of R and S using a composition operator would generate a new SFS, say $T \in SFS(U)$, denoted by:

$$S \circledast R = T; \text{ where } \sim \text{ is any composition operator.}$$

Consequently, we propose the definition of ESFS as follows:

Definition 3 (ESFS). *Let R be a spherical fuzzy relation on a spherical fuzzy set $X \in SFS(U)$, i.e., $R \in SFR(X \times X)$. A SFS $S \in SFS(U)$ is said to be an ESFS associated with the relation R if $S = \{x, \mu_S(x), \eta_S(x), \nu_S(x) \mid x \in X\}$ satisfies the condition $S \circledast R = S$ with $\mu_S(x), \eta_S(x), \nu_S(x) \in [0,1]$.*

Further, the methods for finding the GESFS and the LESFS associated with the spherical fuzzy relation R are outlined.

5.1. Algorithms for finding GESFS

To obtain the GESFS associated with the spherical fuzzy relation R, we apply the $max - min$ composition operator to spherical fuzzy relations.

Let S_1 be the SFS, i.e., $S_1 \in SFS(U)$, in which the membership value is the greatest of all the elements of the column of relation R and the neutral membership and the non-membership values are the smallest of all the elements of the columns of R, i.e.,

$$\mu_{S_1}(x') = \max_{x \in X} \mu_R(x, x'), \text{ for all } x' \in Y,$$

$$\eta_{S_1}(x') = \min_{x \in X} \eta_R(x, x'), \text{ for all } x' \in Y,$$

$$\nu_{S_1}(x') = \min_{x \in X} \nu_R(x, x'), \text{ for all } x' \in Y. \quad (1)$$

The process is initiated by taking S_0 as a constant SFS with a value equal to the minimum element of the set S_1. It is easy to verify that S_0 is an ESFS, but not the GESFS always. To overcome this difficulty, we define the following sequence of SFS S_n such that:

$$S_1 \circ R = S_2$$

$$S_2 \circ R = S_1 \circ R^2 = S_3$$

$$\vdots \qquad \vdots$$

$$S_n \circ R = S_1 \circ R^n = S_{n+1}$$

It may be observed that the obtained sequence S_n is a decreasing sequence and bounded by S_0 and S_1, i.e., $S_0 \subseteq \ldots \subseteq S_{n+1} \subseteq S_n \subseteq \ldots \subseteq S_3 \subseteq S_2 \subseteq S_1$. Next, we present two fundamental algorithms along with a numerical example to determine GESFS as follows:

Algorithm I (GESFS):

- **Step 1.** Find the set S_1 from R as directed by Eq. (1);
- **Step 2.** Set the index $n = 1$ and calculate $S_{n+1} = S_n \circ R$;
- **Step 3.** If $S_{n+1} \neq S_n$, then return to Step 2;
- **Step 4.** If $S_{n+1} = S_n$, then S_n is the GESFS associated with R.

The proposed algorithm is presented using the flow chart given in Figure 2.

Example 1. *Let $X = \{x_1, x_2, x_3, x_4\}$ be a SFS and R be the spherical fuzzy relation on X represented by equation shown in Box I.*

The computational steps to find the GESFS are as follows:

- **Step 1.** The set S_1 is given by:

$$S_1 = [(0.9, 0.1, 0.1), (0.8, 0.1, 0.2),$$
$$(0.9, 0.1, 0.1), (0.8, 0.1, 0.1)].$$

- **Step 2.** For $n = 1$, we have the relation shown in Box II.
- **Step 3.** Since $S_2 \neq S_1$, we set $n = 2$ in Step 2 and compose S_2 with R to get S_3; this is shown in Box III.

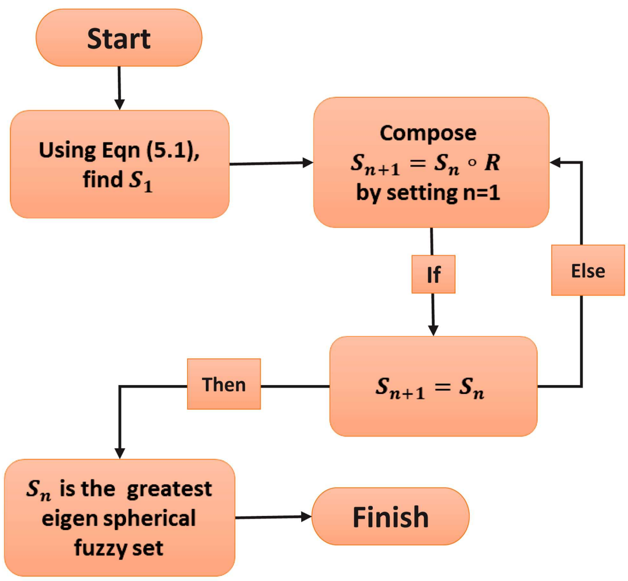

Figure 2. Flow chart for Algorithm I (GESFS).

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad x_1 \quad & \quad x_2 \quad & \quad x_3 \quad & \quad x_4 \quad \end{array} \\ \begin{pmatrix} (0.7,0.1,0.2) & (0.8,0.1,0.2) & (0.5,0.3,0.4) & (0.8,0.1,0.1) \\ (0.9,0.1,0.2) & (0.5,0.4,0.2) & (0.8,0.2,0.2) & (0.6,0.6,0.1) \\ (0.5,0.4,0.3) & (0.6,0.2,0.3) & (0.9,0.1,0.3) & (0.5,0.4,0.2) \\ (0.7,0.5,0.1) & (0.4,0.5,0.2) & (0.6,0.4,0.1) & (0.2,0.6,0.4) \end{pmatrix} \end{array}.$$

Box I

$$S_2 = S_1 \circ R$$

$$= \begin{bmatrix} (0.9,0.1,0.1) & (0.8,0.1,0.2) & (0.9,0.1,0.1) & (0.8,0.1,0.1) \end{bmatrix} \circ \begin{bmatrix} (0.7,0.1,0.2) & (0.8,0.1,0.2) & (0.5,0.3,0.4) & (0.8,0.1,0.1) \\ (0.9,0.1,0.2) & (0.5,0.4,0.2) & (0.8,0.2,0.2) & (0.6,0.6,0.1) \\ (0.5,0.4,0.3) & (0.6,0.2,0.3) & (0.9,0.1,0.3) & (0.5,0.4,0.2) \\ (0.7,0.5,0.1) & (0.4,0.5,0.2) & (0.6,0.4,0.1) & (0.2,0.6,0.4) \end{bmatrix}$$

i.e., $S_2 = [(0.8,0.1,0.1),(0.8,0.1,0.2),(0.9,0.1,0.1),(0.8,0.1,0.1)]$.

Box II

- **Step 4.** Since $S_3 = S_2$, S_2 is the GESFS associated with R.

Algorithm II (GESFS):

- **Step 1.** Find the set S_1 from R as directed by Eq. (1),
- **Step 2.** Using the successive composition of R, say, $R^{n+1} = \underbrace{R \circ R \circ R \ldots \circ R}_{n+1}$, compute S_{n+1} from R^{n+1} using Eq. (1),
- **Step 3.** If $S_{n+1} \neq S_n$, then return to Step 2,
- **Step 4.** If $S_{n+1} = S_n$, then S_n is the GESFS associated with the relation R.

We consider the same example, i.e., Example 1, for the illustration of the computational steps of Algorithm II as below:

- **Step 1.** Using Eq. (1), the set S_1 is given by:

$$S_1 = [(0.9,0.1,0.1),(0.8,0.1,0.2),$$
$$(0.9,0.1,0.1),(0.8,0.1,0.1)]$$

- **Step 2.** To find S_2, we compose R with itself; the relations are shown in Box IV. Therefore,

$$S_2 = [(0.7,0.1,0.1),(0.8,0.1,0.2),(0.9,0.1,0.1),$$
$$(0.8,0.1,0.1)].$$

$$S_3 = S_2 \circ R$$

$$= \begin{bmatrix} (0.8,0.1,0.1) & (0.8,0.1,0.1) & (0.9,0.1,0.1) & (0.8,0.1,0.1) \end{bmatrix} \circ \begin{bmatrix} (0.7,0.1,0.2) & (0.8,0.1,0.2) & (0.5,0.3,0.4) & (0.8,0.1,0.1) \\ (0.9,0.1,0.2) & (0.5,0.4,0.2) & (0.8,0.2,0.2) & (0.6,0.6,0.1) \\ (0.5,0.4,0.3) & (0.6,0.2,0.3) & (0.9,0.1,0.3) & (0.5,0.4,0.2) \\ (0.7,0.5,0.1) & (0.4,0.5,0.2) & (0.6,0.4,0.1) & (0.2,0.6,0.4) \end{bmatrix}$$

i.e., $S_3 = [(0.8,0.1,0.1),(0.8,0.1,0.2),(0.9,0.1,0.1),(0.8,0.1,0.1)]$.

Box III

$$R^2 = R \circ R$$

$$= \begin{bmatrix} (0.7,0.1,0.2) & (0.8,0.1,0.2) & (0.5,0.3,0.4) & (0.8,0.1,0.1) \\ (0.9,0.1,0.2) & (0.5,0.4,0.2) & (0.8,0.2,0.2) & (0.6,0.6,0.1) \\ (0.5,0.4,0.3) & (0.6,0.2,0.3) & (0.9,0.1,0.3) & (0.5,0.4,0.2) \\ (0.7,0.5,0.1) & (0.4,0.5,0.2) & (0.6,0.4,0.1) & (0.2,0.6,0.4) \end{bmatrix} \circ \begin{bmatrix} (0.7,0.1,0.2) & (0.8,0.1,0.2) & (0.5,0.3,0.4) & (0.8,0.1,0.1) \\ (0.9,0.1,0.2) & (0.5,0.4,0.2) & (0.8,0.2,0.2) & (0.6,0.6,0.1) \\ (0.5,0.4,0.3) & (0.6,0.2,0.3) & (0.9,0.1,0.3) & (0.5,0.4,0.2) \\ (0.7,0.5,0.1) & (0.4,0.5,0.2) & (0.6,0.4,0.1) & (0.2,0.6,0.4) \end{bmatrix}$$

$$R^2 = \begin{bmatrix} (0.8,0.1,0.1) & (0.7,0.1,0.2) & (0.8,0.1,0.1) & (0.7,0.1,0.2) \\ (0.7,0.1,0.1) & (0.8,0.1,0.2) & (0.8,0.1,0.1) & (0.8,0.1,0.2) \\ (0.6,0.1,0.2) & (0.6,0.1,0.2) & (0.9,0.1,0.2) & (0.6,0.1,0.3) \\ (0.7,0.1,0.2) & (0.7,0.1,0.2) & (0.6,0.1,0.2) & (0.7,0.1,0.1) \end{bmatrix}.$$

Box IV

$$R^3 = R^2 \circ R$$

$$= \begin{bmatrix} (0.8,0.1,0.1) & (0.7,0.1,0.2) & (0.8,0.1,0.1) & (0.7,0.1,0.2) \\ (0.7,0.1,0.1) & (0.8,0.1,0.2) & (0.8,0.1,0.1) & (0.8,0.1,0.2) \\ (0.6,0.1,0.2) & (0.6,0.1,0.2) & (0.9,0.1,0.2) & (0.6,0.1,0.3) \\ (0.7,0.1,0.2) & (0.7,0.1,0.2) & (0.6,0.1,0.2) & (0.7,0.1,0.1) \end{bmatrix} \circ \begin{bmatrix} (0.7,0.1,0.2) & (0.8,0.1,0.2) & (0.5,0.3,0.4) & (0.8,0.1,0.1) \\ (0.9,0.1,0.2) & (0.5,0.4,0.2) & (0.8,0.2,0.2) & (0.6,0.6,0.1) \\ (0.5,0.4,0.3) & (0.6,0.2,0.3) & (0.9,0.1,0.3) & (0.5,0.4,0.2) \\ (0.7,0.5,0.1) & (0.4,0.5,0.2) & (0.6,0.4,0.1) & (0.2,0.6,0.4) \end{bmatrix}$$

$$R^3 = \begin{bmatrix} (0.7,0.1,0.2) & (0.8,0.1,0.2) & (0.8,0.1,0.2) & (0.8,0.1,0.1) \\ (0.8,0.1,0.2) & (0.7,0.1,0.2) & (0.8,0.1,0.2) & (0.7,0.1,0.1) \\ (0.6,0.1,0.2) & (0.6,0.1,0.2) & (0.9,0.1,0.2) & (0.6,0.1,0.2) \\ (0.7,0.1,0.1) & (0.7,0.1,0.2) & (0.7,0.1,0.1) & (0.7,0.1,0.2) \end{bmatrix}.$$

Box V

- **Step 3.** Since $S_2 \neq S_1$, we find S_3 by further composing R^2 with R; the relations are shown in Box V. Therefore,

$$S_3 = [(0.8, 0.1, 0.1), (0.8, 0.1, 0.2), (0.9, 0.1, 0.1),$$
$$(0.8, 0.1, 0.1)].$$

- **Step 4.** Since $S_3 = S_2$, S_2 is the GESFS associated with R.

5.2. Algorithms for finding LESFS

To obtain the LESFS associated with the spherical fuzzy relation R, we apply the $min - max$ composition operator to spherical fuzzy relations.

Let S_1 be the SFS, i.e., $S_1 \in SFS(U)$ in which the membership and neutral membership values are the smallest of all the elements of the column of relation R and the non-membership value is the greatest of all the elements of the columns of R, i.e.,

$$\mu_{S_1}(x') = \min_{x \in X} \mu_R(x, x'), \text{ for all } x' \in Y,$$

$$\eta_{S_1}(x') = \min_{x \in X} \eta_R(x, x'), \text{ for all } x' \in Y,$$

$$\nu_{S_1}(x') = \max_{x \in X} \nu_R(x, x'), \text{ for all } x' \in Y. \tag{2}$$

Here, we initiate the process by taking S_0 as a constant SFS with a value equal to the minimum element of the set S_1. It is easy to verify that S_0 is an ESFS, but not the LESFS always. To overcome this difficulty, we define the following sequence of SFS S_n such that:

$$S_1 \bullet R = S_2$$

$$S_2 \bullet R = S_1 \bullet R^2 = S_3$$

$$\vdots \quad \vdots$$

$$S_n \bullet R = S_1 \bullet R^n = S_{n+1}$$

Next, we present two fundamental algorithms along with a numerical example for the determination of the LESFS as follows:

Algorithm I (LESFS):

- **Step 1.** Find the set S_1 as directed by Eq. (2);
- **Step 2.** Set the index $n = 1$ and calculate $S_{n+1} = S_n \bullet R$;
- **Step 3.** If $S_{n+1} \neq S_n$, then return to Step 2;
- **Step 4.** If $S_{n+1} = S_n$, then S_n is the LESFS associated with R.

We consider the same example again, i.e., Example 1, for the illustration of the computational steps of Algorithm I (LESFS) as follows:

- **Step 1.** The set S_1 is given by:

$$S_1 = [(0.5, 0.1, 0.3), (0.4, 0.1, 0.3),$$
$$(0.5, 0.1, 0.4), (0.2, 0.1, 0.4)].$$

- **Step 2.** For $n = 1$, we have the equations shown in Box VI.

$$S_2 = S_1 \bullet R$$

$$= \left[(0.5,0.1,0.3),(0.4,0.1,0.3),(0.5,0.1,0.4),(0.2,0.1,0.4)\right] \bullet \begin{bmatrix} (0.7,0.1,0.2) & (0.8,0.1,0.2) & (0.5,0.3,0.4) & (0.8,0.1,0.1) \\ (0.9,0.1,0.2) & (0.5,0.4,0.2) & (0.8,0.2,0.2) & (0.6,0.6,0.1) \\ (0.5,0.4,0.3) & (0.6,0.2,0.3) & (0.9,0.1,0.3) & (0.5,0.4,0.2) \\ (0.7,0.5,0.1) & (0.4,0.5,0.2) & (0.6,0.4,0.1) & (0.2,0.6,0.4) \end{bmatrix}$$

i.e., $S_2 = \left[(0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4)\right].$

Box VI

$$S_3 = S_2 \bullet R$$

$$= \left[(0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4)\right] \bullet \begin{bmatrix} (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\ (0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\ (0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\ (0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4) \end{bmatrix}$$

i.e., $S_3 = \left[(0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4)\right]$.

Box VII

- **Step 3.** Since $S_2 \neq S_1$, we set $n = 2$ in Step 2 and compose S_2 with R to get S_3; the relations are shown in Box VII.
- **Step 4.** Since $S_3 = S_2$, S_2 is the LESFS associated with R.

Algorithm II (LESFS):

- **Step 1.** Find the set S_1 from R as directed by Eq. (2).
- **Step 2.** Using the successive composition of R, say, $R^{n+1} = \underbrace{R \bullet R \bullet R \ldots \bullet R}_{n+1}$, compute S_{n+1} from R^{n+1} using Eq. (2).
- **Step 3.** If $S_{n+1} \neq S_n$, then return to Step 2.
- **Step 4.** If $S_{n+1} = S_n$, then S_n is the LESFS associated with the relation R.

The proposed algorithm is presented using the flow chart given in Figure 3.

We consider the same example, i.e., Example 1, for the illustration of the computational steps of Algorithm II (LESFS) as follows:

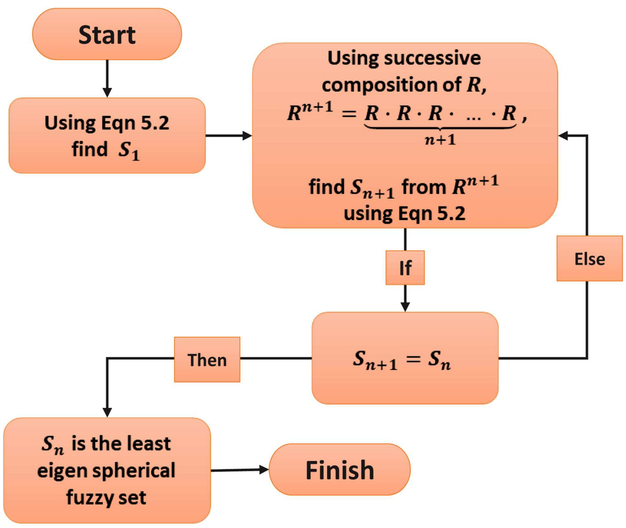

Figure 3. Flow chart for Algorithm II (LESFS).

- **Step 1.** Using Eq. (2), the set S_1 is given by:

$$S_1 = [(0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.4), (0.2, 0.1, 0.4)].$$

- **Step 2.** To find S_2, we compose R with itself; the relations are shown in Box VIII. Therefore:

$$S_2 = [(0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4)].$$

- **Step 3.** Since $S_2 \neq S_1$, we find S_3 by further composing R^2 with R; the relations are shown in Box IX. Therefore,

$$S_3 = [(0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4)].$$

- **Step 4.** Since $S_3 = S_2$, S_2 is the LESFS associated with R.

6. Multi-criteria decision-making using GESFS and LESFS

In a decision-making problem, because of fuzziness in human thinking, there is always a kind of complexity and uncertainty which is inherent in the available alternatives and laid down criteria. Therefore, it is difficult to evaluate the parameters of the decision process with the desired preciseness. Therefore, the problem of multicriteria decision-making has been widely dealt with and utilized in various applications [17,18,31–33]. On the basis of GESFS and LESFS, similar applications in the above-stated field of decision-making may also be studied. However, to illustrate the proposed methodology, the following examples are considered.

6.1. Example related to insurance company

We consider an insurance company where the satisfaction/abstention/non-satisfaction levels of the customers are taken into account for formulating the multiple-criteria decision-making problem.

$$R^2 = R \bullet R$$

$$= \begin{bmatrix} (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\ (0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\ (0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\ (0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4) \end{bmatrix} \bullet \begin{bmatrix} (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\ (0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\ (0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\ (0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4) \end{bmatrix}$$

$$R^2 = \begin{bmatrix} (0.5, 0.1, 0.3) & (0.6, 0.1, 0.3) & (0.7, 0.1, 0.3) & (0.5, 0.1, 0.2) \\ (0.7, 0.1, 0.2) & (0.5, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.6, 0.1, 0.2) \\ (0.7, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.2) \\ (0.6, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.2, 0.1, 0.4) \end{bmatrix}.$$

Box VIII

$$R^3 = R^2 \bullet R$$

$$= \begin{bmatrix} (0.5, 0.1, 0.3) & (0.6, 0.1, 0.3) & (0.7, 0.1, 0.3) & (0.5, 0.1, 0.2) \\ (0.7, 0.1, 0.2) & (0.5, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.6, 0.1, 0.2) \\ (0.7, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.2) \\ (0.6, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.2, 0.1, 0.4) \end{bmatrix} \bullet \begin{bmatrix} (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\ (0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\ (0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\ (0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4) \end{bmatrix}$$

$$R^3 = \begin{bmatrix} (0.7, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.2) \\ (0.6, 0.1, 0.2) & (0.5, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.6, 0.1, 0.2) \\ (0.5, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.6, 0.1, 0.3) & (0.5, 0.1, 0.2) \\ (0.6, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.2, 0.1, 0.4) \end{bmatrix}.$$

Box IX

Example 2 *Suppose that an insurance company collects the information from 10 of its prime customers/experts about the important features of the company. Let the features be listed as:*

- F_1 : Customer friendly policies;
- F_2 : Amplitude of financial benefits;
- F_3 : Post insurance services.

We may figure out the customer's feedback on the basis of a survey. However, in order to illustrate the proposed methodology, we assume a set of data presented below without an exhaustive survey.

For evaluating some concluding remarks in view of the insurance company, we assume each customer's feedback as spherical fuzzy information in a relative fashion among all the available/provided features and tabulate them in Tables 1–3.

The estimation of the desired levels in the form of the satisfaction/abstention/non-satisfaction levels is possible by considering the spherical fuzzy relation. Each pair of the relation $R_{(F_j,F_k)}$ has the membership value (satisfied), the indeterminacy value (abstention), and the non-membership value (not satisfied) which range from 0 to 1 as given below:

$$R_{(F_j,F_k)} = \left(\frac{\sum\limits_{p=1,q=1}^{p=m,q=n} \mu_{pq}}{m}, \frac{\sum\limits_{p=1,q=1}^{p=m,q=n} \eta_{pq}}{m}, \frac{\sum\limits_{p=1,q=1}^{p=m,q=n} \nu_{pq}}{m} \right), \quad (3)$$

Table 1. Relative feedback with F_1 and F_2.

Customers/experts	$\boldsymbol{F_1}$	$\boldsymbol{F_2}$
E_1	$(0.5, 0.1, 0.6)$	$(0.8, 0.3, 0.2)$
E_2	$(0.6, 0.6, 0.1)$	$(0.7, 0.6, 0.2)$
E_3	$(0.7, 0.3, 0.2)$	$(0.9, 0.2, 0.1)$
E_4	$(0.8, 0.1, 0.2)$	$(0.6, 0.6, 0.1)$
E_5	$(0.4, 0.6, 0.1)$	$(0.5, 0.4, 0.4)$
E_6	$(0.3, 0.4, 0.4)$	$(0.5, 0.5, 0.2)$
E_7	$(0.2, 0.4, 0.6)$	$(0.4, 0.2, 0.6)$
E_8	$(0.8, 0.2, 0.1)$	$(0.3, 0.2, 0.7)$
E_9	$(0.3, 0.2, 0.7)$	$(0.8, 0.3, 0.2)$
E_{10}	$(0.9, 0.1, 0.2)$	$(0.4, 0.4, 0.4)$

Table 2. Relative feedback with F_1 and F_3.

Customers/experts	$\boldsymbol{F_1}$	$\boldsymbol{F_3}$
E_1	$(0.7, 0.3, 0.2)$	$(0.5, 0.5, 0.1)$
E_2	$(0.4, 0.5, 0.4)$	$(0.4, 0.2, 0.6)$
E_3	$(0.6, 0.3, 0.2)$	$(0.7, 0.3, 0.1)$
E_4	$(0.8, 0.1, 0.2)$	$(0.8, 0.4, 0.1)$
E_5	$(0.4, 0.4, 0.3)$	$(0.9, 0.1, 0.2)$
E_6	$(0.9, 0.2, 0.1)$	$(0.4, 0.2, 0.6)$
E_7	$(0.6, 0.3, 0.2)$	$(0.2, 0.1, 0.7)$
E_8	$(0.7, 0.4, 0.1)$	$(0.6, 0.2, 0.4)$
E_9	$(0.4, 0.1, 0.6)$	$(0.7, 0.1, 0.3)$
E_{10}	$(0.7, 0.2, 0.2)$	$(0.6, 0.3, 0.1)$

Table 3. Relative feedback with F_2 and F_3.

Customers/experts	$\boldsymbol{F_2}$	$\boldsymbol{F_3}$
E_1	$(0.4, 0.5, 0.1)$	$(0.9, 0.1, 0.2)$
E_2	$(0.6, 0.2, 0.3)$	$(0.8, 0.1, 0.1)$
E_3	$(0.8, 0.1, 0.1)$	$(0.7, 0.3, 0.1)$
E_4	$(0.9, 0.1, 0.2)$	$(0.6, 0.4, 0.1)$
E_5	$(0.7, 0.4, 0.1)$	$(0.5, 0.1, 0.5)$
E_6	$(0.8, 0.1, 0.3)$	$(0.4, 0.3, 0.1)$
E_7	$(0.4, 0.4, 0.3)$	$(0.6, 0.4, 0.1)$
E_8	$(0.2, 0.7, 0.1)$	$(0.4, 0.2, 0.4)$
E_9	$(0.9, 0.1, 0.1)$	$(0.5, 0.5, 0.2)$
E_{10}	$(0.5, 0.1, 0.4)$	$(0.6, 0.4, 0.2)$

and:

$$R_{(F_i,F_i)} = \frac{R_{(F_i,F_j)} + R_{(F_i,F_k)}}{2}, \tag{4}$$

where $j,\ k = 1, 2, \ldots, n$.

Using Eq. (3) and Eq. (4), the membership, indeterminacy, and non-membership values for the different pairs of features are computed as follows:

$$R_{(F_1,F_1)} = (0.585, 0.29, 0.285),$$

$$R_{(F_1,F_2)} = (0.55, 0.30, 0.32),$$

$$R_{(F_1,F_3)} = (0.62, 0.28, 0.25),$$

$$R_{(F_2,F_1)} = (0.59, 0.37, 0.31),$$

$$R_{(F_2,F_2)} = (0.605, 0.32, 0.255),$$

$$R_{(F_2,F_3)} = (0.62, 0.27, 0.20),$$

$$R_{(F_3,F_1)} = (0.58, 0.24, 0.32),$$

$$R_{(F_3,F_2)} = (0.60, 0.28, 0.20),$$

$$R_{(F_3,F_3)} = (0.59, 0.26, 0.26).$$

Next, we construct the spherical fuzzy relation R using the above-obtained inter-dependency of the features as follows:

$$R = \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \end{array} \begin{array}{c} \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{pmatrix} R_{(F_1,F_1)} & R_{(F_1,F_2)} & R_{(F_1,F_3)} \\ R_{(F_2,F_1)} & R_{(F_2,F_2)} & R_{(F_2,F_3)} \\ R_{(F_3,F_1)} & R_{(F_3,F_2)} & R_{(F_3,F_3)} \end{pmatrix} \end{array}.$$

Setting all the values, we obtain the equation shown in Box X.

Now, we use the first proposed algorithm for finding the GESFS, i.e., Algorithm I (GESFS), and get:

$$S_1 = [(0.59, 0.24, 0.285), (0.605, 0.28, 0.2), (0.62, 0.26, 0.2)]],$$

$$S_2 = S_1 \circ R = (0.59, 0.24, 0.285), (0.605, 0.24, 0.2), (0.605, 0.24, 0.2)],$$

$$S_3 = S_2 \circ R = [(0.59, 0.24, 0.285), (0.605, 0.24, 0.2), (0.605, 0.24, 0.2)].$$

Since $S_2 = S_3$, we conclude that S_2 is the GESFS.

Further, we use the first proposed algorithm for finding the LESFS, i.e., Algorithm I (LESFS), and get:

$$S_1 = (0.58, 0.24, 0.32), (0.55, 0.28, 0.32), (0.59, 0.26, 0.26)],$$

$$S_2 = S_1 \bullet R = [(0.585, 0.24, 0.31), (0.58, 0.24, 0.32), (0.59, 0.24, 0.26)],$$

$$S_3 = S_2 \bullet R = [(0.585, 0.24, 0.31), (0.585, 0.24, 0.31), (0.59, 0.24, 0.26)],$$

$$S_4 = S_3 \bullet R = [(0.585, 0.24, 0.31), (0.585, 0.24, 0.31), (0.59, 0.24, 0.26)].$$

Since $S_3 = S_4$, we conclude that S_3 is the LESFS.

Observations and results

On the basis of computations, we have found that the GESFS and the LESFS are respectively given by:

$$R = \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \end{array} \begin{array}{c} \begin{array}{ccc} F_1 & F_2 & F_3 \end{array} \\ \begin{pmatrix} (0.585, 0.29, 0.285) & (0.55, 0.30, 0.32) & (0.62, 0.28, 0.25) \\ (0.59, 0.37, 0.31) & (0.605, 0.32, 0.255) & (0.62, 0.27, 0.20) \\ (0.58, 0.24, 0.32) & (0.60, 0.28, 0.20) & (0.59, 0.26, 0.26) \end{pmatrix} \end{array}.$$

Box X

$$GESFS = [(0.59, 0.24, 0.285), (0.605, 0.24, 0.2), (0.605, 0.24, 0.2)],$$

and:

$$LESFS = [(0.585, 0.24, 0.31), (0.585, 0.24, 0.31), (0.59, 0.24, 0.26)].$$

The values obtained from these sets represent the range of satisfaction/abstention/non-satisfaction levels for the features under consideration for an insurance company:

- Customers are satisfied in the range of 58.5% to 59%, abstain (24%), and unsatisfied in the range of 28.5% to 31% with respect to feature F_1;
- Customers are satisfied in the range of 58.5% to 60.5%, abstain (24%), and unsatisfied in the range of 20% to 31% with respect to feature F_2;
- Customers are satisfied in the range of 59% to 60.5%, abstain (24%), and unsatisfied in the range of 20% to 26% with respect to feature F_3.

It may be noted that the numerical values obtained from the GESFS and LESFS are close to each other. In fact, the illustration of the proposed algorithms has been done through the particular example (Example 2) which has a limited format and less variability with respect to the dimensions and features involved in it. If we will have a big data with higher dimensionality of features, we may observe a significant variation in the values. However, the closeness in the values is a sign of preciseness of the process of decision-making.

6.2. Example related to E-learning websites

Recently, Garg et al. [34] proposed a decision support system to educational organizations to develop and access E-learning websites employing a hybrid multi-attribute decision-making method for their evaluation and ranking. Jain et al. [35] utilized the weighted distance-based approximation for selection and ranking of E-websites. Based on weighted Euclidean distance, Garg [36] proposed a computational quantitative model in order to evaluate, select, and rank E-learning websites. Garg and Arora [37] evaluated fraud detection model versus selection criteria as a Multi-Criteria Decision-Making (MCDM) problem and obtained a validated comprehensive ranking.

We consider three E-learning websites where the satisfaction/abstention/non-satisfaction levels of the users are taken into account for formulating the decision-making problem. We assume each user's feedback as spherical fuzzy information and tabulate the performance ratings of E-learning websites in Tables 4–6.

Table 4. Relative feedback with W_1 and W_2.

Customers/experts	W_1	W_2
U_1	$(0.7, 0.1, 0.6)$	$(0.8, 0.2, 0.1)$
U_2	$(0.8, 0.2, 0.1)$	$(0.3, 0.2, 0.7)$
U_3	$(0.7, 0.3, 0.2)$	$(0.9, 0.2, 0.1)$
U_4	$(0.9, 0.1, 0.2)$	$(0.4, 0.2, 0.6)$
U_5	$(0.4, 0.6, 0.1)$	$(0.5, 0.4, 0.4)$
U_6	$(0.6, 0.6, 0.1)$	$(0.4, 0.6, 0.1)$
U_7	$(0.7, 0.3, 0.2)$	$(0.8, 0.3, 0.2)$
U_8	$(0.6, 0.2, 0.5)$	$(0.4, 0.5, 0.4)$
U_9	$(0.2, 0.4, 0.6)$	$(0.4, 0.2, 0.6)$
U_{10}	$(0.9, 0.2, 0.3)$	$(0.3, 0.2, 0.5)$

Table 5. Relative feedback with W_1 and W_3.

Customers/experts	W_1	W_3
U_1	$(0.6, 0.3, 0.2)$	$(0.7, 0.3, 0.2)$
U_2	$(0.7, 0.2, 0.3)$	$(0.5, 0.5, 0.2)$
U_3	$(0.9, 0.1, 0.1)$	$(0.3, 0.6, 0.2)$
U_4	$(0.6, 0.2, 0.3)$	$(0.4, 0.3, 0.6)$
U_5	$(0.4, 0.5, 0.4)$	$(0.6, 0.2, 0.4)$
U_6	$(0.7, 0.2, 0.2)$	$(0.6, 0.3, 0.1)$
U_7	$(0.9, 0.2, 0.1)$	$(0.4, 0.2, 0.6)$
U_8	$(0.4, 0.2, 0.5)$	$(0.6, 0.3, 0.2)$
U_9	$(0.8, 0.1, 0.2)$	$(0.5, 0.2, 0.4)$
U_{10}	$(0.2, 0.2, 0.6)$	$(0.9, 0.1, 0.2)$

Table 6. Relative feedback with W_2 and W_3.

Customers/experts	W_2	W_3
U_1	$(0.9, 0.1, 0.2)$	$(0.6, 0.2, 0.3)$
U_2	$(0.8, 0.4, 0.1)$	$(0.4, 0.5, 0.1)$
U_3	$(0.8, 0.2, 0.3)$	$(0.4, 0.3, 0.1)$
U_4	$(0.2, 0.7, 0.1)$	$(0.8, 0.2, 0.1)$
U_5	$(0.5, 0.5, 0.2)$	$(0.5, 0.4, 0.4)$
U_6	$(0.6, 0.4, 0.2)$	$(0.4, 0.4, 0.3)$
U_7	$(0.2, 0.6, 0.5)$	$(0.9, 0.2, 0.2)$
U_8	$(0.8, 0.2, 0.1)$	$(0.4, 0.5, 0.2)$
U_9	$(0.5, 0.2, 0.4)$	$(0.6, 0.4, 0.1)$
U_{10}	$(0.9, 0.2, 0.1)$	$(0.5, 0.5, 0.3)$

Next, we construct the spherical fuzzy relation R using the above obtained inter-dependency of the features as follows:

$$R = \begin{array}{c} \\ W_1 \\ W_2 \\ W_3 \end{array} \begin{array}{c} \begin{array}{ccc} W_1 & W_2 & W_3 \end{array} \\ \begin{pmatrix} R_{(W_1,W_1)} & R_{(W_1,W_2)} & R_{(W_1,W_3)} \\ R_{(W_2,W_1)} & R_{(W_2,W_2)} & R_{(W_2,W_3)} \\ R_{(W_3,W_1)} & R_{(W_3,W_2)} & R_{(W_3,W_3)} \end{pmatrix} \end{array}.$$

Setting all the values, we obtain the equation shown in Box XI.

$$R = \begin{array}{c} \\ W_1 \\ W_2 \\ W_3 \end{array} \begin{array}{c} \begin{array}{ccc} W_1 & W_2 & W_3 \end{array} \\ \begin{pmatrix} (0.615, 0.255, 0.315) & (0.63, 0.29, 0.33) & (0.62, 0.24, 0.29) \\ (0.56, 0.30, 0.36) & (0.59, 0.325, 0.29) & (0.67, 0.35, 0.22) \\ (0.55, 0.30, 0.31) & (0.26, 0.36, 0.21) & (0.555, 0.33, 0.26) \end{pmatrix} \end{array}.$$

Box XI

Next, on the basis of computations, we have found that the GESFS and the LESFS are respectively given by:

$$GESFS = [(0.615, 0.22, 0.31), (0.63, 0.24, 0.22), (0.67, 0.24, 0.22)],$$

and:

$$LESFS = [(0.555, 0.22, 0.33), (0.56, 0.24, 0.33), (0.555, 0.24, 0.29)].$$

The values obtained in these sets represent the range of satisfaction/abstention/non-satisfaction level among the users for the websites under consideration:

- Users are satisfied in the range of 55.5% to 61.5%, abstain (22%), and unsatisfied in the range of 31% to 33% with respect to the features of website W_1;
- Users are satisfied in the range of 56% to 63%, abstain (22%), and unsatisfied in the range of 22% to 33% with respect to the features of website W_2;
- Users are satisfied in the range of 55.5% to 67%, abstain (24%), and unsatisfied in the range of 22% to 29% with respect to the features of website W_3.

However, the closeness of the obtained values of the range is not significant as the data are hypothetical and quite small in size; however, for a large sample data, the values will certainly characterize the attainment level clearly.

7. Scope for future work

7.1. Image information retrieval

In the field of fuzzy image information analysis, an input original image is considered to be a fuzzy relation by a process of image intensity normalization. We may consider the idea of GESFS with respect to the $max - min$ composition operator and the LESFS with respect to the $min - max$ composition operator using the spherical fuzzy relation for solving the problems of retrieving the image information. A comparison between the sample image and the retrieved images can be done by using the similarity measure based on the GESFS and LESFS as follows:

$$D(R_1, R_2) = \sum_{x \in X} \left((S_1(x) - S_2(x))^2 + (T_1(x) - T_2(x))^2 \right),$$

where $S_i \,\&\, T_i \in SFS(P)$ are the GESFS and LESFS of the relation $R_i \in SFR(P \times P); i = 1, 2$ ($P = \{1, 2, \ldots, 256\}$) with respect to the $max - min$ and $min - max$ composition, respectively.

In literature, it may be noted that the existing conventional algorithms for obtaining the GEFS/LEFS are not capable enough to deal with the image retrieval problems to achieve a solution significantly of good quality. It has also been observed that the use of the eigen fuzzy sets enhances the quality of the reconstructed image more often.

Using the convex combination of $max - min$ and $min - max$ composition operators for eigen fuzzy sets equations, various eigen fuzzy sets may be generated. Many eigen fuzzy set equations can be obtained through the above-stated convex combination, i.e., by changing the value of λ present in the convex combination. The eigen fuzzy set equation is given by:

$$\lambda \cdot (S \circ R) + (1 - \lambda) \cdot (S \bullet R) = S,$$

where λ can be chosen as per the best fit. The detailed and comprehensive study of the image information retrieval can be carried out by implementing the proposed algorithms and technique along with the above-outlined directions.

7.2. Genetic algorithm for image reconstruction

Fuzzy relational calculus for image compression is a natural tool for a genetic algorithm depending on the eigen fuzzy sets for image reconstruction. By normalizing the values of the pixels of any image (of size $m \times m$) with respect to the length of the gray scale used, it can be interpreted as a square fuzzy relation R. Here, the interpretation will be a spherical fuzzy relation. Implementing the genetic algorithm method would lead to a prospective solution to a particular problem on basic data structure problems, e.g., chromosome in reference to the image by applying the recombination operators to these structures. Genetic

Table 7. Need to address the problem in Intuitionistic Fuzzy Sets (IFSs), Pythagorean Fuzzy Sets (PyFSs), and Picture Fuzzy Sets (PFSs).

R	C_1	C_2	C_3	C_4
C_1	$(1.0+0.0+0.0=1)$	$(0.40+0.20+0.69>1)$	$(0.36+0.19+0.79>1)$	$(0.56+0.17+0.62>1)$
C_2	$(0.68+0.20+0.44>1)$	$(1.0+0.0+0.0>1)$	$(0.40+0.24+0.56>1)$	$(0.51+0.29+0.61>1)$
C_3	$(0.76+0.20+0.42>1)$	$(0.54+0.24+0.42>1)$	$(1.0+0.0+0.0>1)$	$(0.48+0.14+0.77>1)$
C_4	$(0.49+0.17+0.68>1)$	$(0.59+0.29+0.53>1)$	$(0.77+0.17+0.38>1)$	$(1.0+0.0+0.0>1)$

algorithms have been frequently seen as optimization functions. However, the range of the problems to which the genetic algorithms have been applied is broad. The genetic algorithm approach may also be used in reconstructing an image by using its GESFS and LESFS of the spherical fuzzy relation in the fitness function of a chromosome. The value of the fitness function is given by:

$$F(R,R_k)=\frac{1}{MSE_{GESFS\leftrightarrow LESFS}(R,R_k)},$$

where $k=1,2,\dots,N$ and the mean squared error is given by:

$$MSE_{GESFS\leftrightarrow LESFS}(R,R_k)=$$

$$\frac{1}{256}\sum_{x\in X}\left((S(x)-S_k(x))^2+(T(x)-T_k(x))^2\right),$$

where S and $T\in SFS(P)$ are the GESFS and LESFS of the spherical fuzzy relation R, respectively, obtained by normalizing the pixels of the input original image ($P=\{1,2,\dots,256\}$).

In particular, we denote S_k and $T_k\in SFS(P)$ as the GESFS and LESFS, of the spherical fuzzy relation R_k, respectively, obtained by normalizing the pixels of the kth image-chromosome of the population (N).

7.3. *Notion of eigen spherical fuzzy soft sets/soft matrices*

In literature, a variety of extensions of soft sets [38] to imprecise and incomplete information have been proposed. In view of the generalizations and extensions of fuzzy sets shown by Figure 1 in the introduction section, we may further propose extending the notion of ESFS to another new concept of ESFS and eigen spherical complex fuzzy soft set (refer to [39–41]). Naim and Serdar [42] introduced the concept of soft matrices from the soft sets and on similar lines, one can study eigen spherical fuzzy soft matrices and the corresponding various properties in future. Since there is a type of parametrization tool involved in the soft sets and, consequently, in soft matrices, various related applications, e.g., stock management [43], valuation of assets [44], medical diagnosis [45], MCDM [46–49], and dimensionality reduction [50] have been studied recently. Hence, introducing the concept of eigen spherical fuzzy soft sets/soft matrices can lead to a new dimension in the extension of soft set theory and related applications.

8. Comparative remarks and advantages of ESFSs

The proposed notion of ESFS is a novel concept and an advanced extension of the classical fuzzy set. The ESFS have an added advantage to deal with a wider sense of applicability in uncertain situations. In detail, some important comparative remarks and advantages of utilizing ESFS are listed below:

- The existing, IFS, and PFSs are subject to some limitations which make them unable to capture the full information specification, i.e., there is a missing additional component of degree of refusal which is addressed by the SFS;
- When uncertain or imprecise information takes the form of a fuzzy relation then to ensure a kind of invariability in the relation after subsequent transitions, we utilize the concept of eigen fuzzy sets in natural sciences for therapeutic recommendations;
- The drawback in the existing literature of the eigen fuzzy sets is that the condition does not allow the experts/decision-makers to allocate the membership values of their own choice (refer to Table 7). Somehow, this makes decision-makers bounded for providing their input in a particular domain. However, the proposed ESFS provide a generalization feature, which makes a strong impact;
- The discussion over implementing the ESFS and the methodology proposed for insurance problem and E-learning websites problem in Section 6 shows that the proposed work handles the generalized framework effective and consistently. In other words, GESFSs and LESFSs of the spherical fuzzy relations have been employed to approximate the optimal level.

9. Conclusions

This study successfully introduced the concept of spherical fuzzy relation and various composition operators ($max-min$, $min-max$, $max-min$ average, and $min-max$ average) based on the combination of triangular norm and conorm. Further, the formal definition of an eigen spherical fuzzy set of spherical fuzzy relation was provided followed by two respective algorithms to determine the greatest eigen spherical fuzzy sets and least eigen spherical fuzzy sets using the $max-min$ and $min-max$ composition operators. Some numerical examples were also included to illustrate the proposed algorithms. Utilization of the greatest eigen spherical fuzzy sets and least eigen spherical fuzzy sets in the field of decision-making problem was also successfully presented. The proposed algorithms could also be applied to a dataset with more variability as well. The directions for future work in the field of image information retrieval as well as genetic algorithm for image reconstruction and outlines to introduce the notion of eigen spherical fuzzy soft sets/soft matrices were briefly stated for further research.

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