CHAPTER

11 SHRINKAGE ESTIMATORS OF SCALE PARAMETER TOWARDS AN INTERVAL OF MORGENSTERN TYPE BIVARIATE UNIFORM DISTRIBUTION USING RANKED SET SAMPLING

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11.1 INTRODUCTION

Ranked set sampling (RSS) is a method of sampling that can be advantageous when quantification of all sampling units is costly but a small set of units can be easily ranked, according to the character under investigation, without actual quantification. The technique was first introduced by McIntyre (1952) for estimating mean pasture and forage yields. The theory and applications of RSS are given by Chen et al. (2004). Suppose the variable of interest, Y, is difficult or much too expensive to measure, but an auxiliary variable X correlated with Y is readily measureable and can be ordered exactly. In this case, as an alternative to McIntyre's (1952) method of ranked set sampling, Stokes (1977) used an auxiliary variable for the ranking of sampling units. If $X_{(r)r}$ is the observation measured on the auxiliary variable X from the unit chosen from the rth set then we write $Y_{[r]r}$ to denote the corresponding measurement made on the study variable Y on this unit, then $Y_{[r]r}, r = 1, 2, \ldots, n$, from the ranked set sample. Clearly, $Y_{[r]r}$ is the concomitant of the rth order statistic arising from the rth sample. Stokes (1995) has obtained the estimation of parameters of the location-scale family of distribution by RSS. Lam et al. (1994) used RSS to estimate the two-parameter exponential distribution. Al-Saleh and Ananbeh (2005, 2007) estimated the means of the bivariate normal distribution using moving extremes RSS with a concomitant variable. Al-Saleh and Diab (2009) considered estimation of the parameters of Downton's bivariate exponential distribution using an RSS scheme. Barnett and Moore (1997) derived the best linear unbiased estimator (BLUE) for the mean of Y, based on a ranked set sample obtained using an auxiliary variable X for ranking the sample units.

In the estimation of an unknown parameter there often exists some prior knowledge about the parameter which one would like to utilize in order to get a better estimate. The Bayesian approach is a well-known example in which prior knowledge about the parameter is available in the form of

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prior distribution. For current references in this context the reader is referred to Sharma et al. (2016), Bouza (2001, 2002, 2005), Samawi and Muttlak (1996), Demir and Singh (2000), Singh and Mehta (2013, 2014a,b, 2015, 2016a,b,c, 2017), Mehta and Singh (2015, 2014), and Mehta (2017).

The organization of this chapter is as follows. Section 11.2 introduces the general distribution theory, properties of Farlie–Gumbel–Morgenstern (FGM) distribution/Morgenstern distribution and a brief review of the estimators of the scale parameter θ_2 envisaged by Tahmasebi and Jafari (2012). In Section 11.3, some improved shrinkage toward interval estimators are described on the lines of Singh et al. (1973), Searls and Intarapanich (1960), Searls (1964), Jani (1991), and Kourouklis (1994), the expressions of bias and mean squared error (MSE) are obtained and compared with usual unbiased estimators. In Section 11.4, we have computed the relative efficiencies of different estimators numerically to evaluate their performance. Section 11.5 concludes the chapter with some final remarks.

11.2 REVIEW OF RSS IN FGM FAMILY OF DISTRIBUTION

A general family of bivariate distributions is proposed by Morgenstern (1956) with specified marginal distributions $F_X(x)$ and $F_Y(y)$ as

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\alpha(1-F_X(x))\,(1-F_Y(y))];\ -1\le\alpha\le 1, \tag{11.1}$$

where α is the association parameter between X and Y.

A member of this family is Morgenstern type bivariate uniform distribution (MTBUD) with the probability density function (pdf)

$$f_{X,Y}(x,y) = \frac{1}{\theta_1\theta_2}\left[1+\alpha\left(1-\frac{2x}{\theta_1}\right)\left(1-\frac{2y}{\theta_2}\right)\right];0<x<\theta_1, 0<y<\theta_2. \tag{11.2}$$

The pdf of $Y_{[r]r}$ for $1\le r\le n$ is given by [see Scaria and Nair (1999)]

$$g_{Y_{[r]r}}(y) = \int f_{Y|X}(y|x)\,f_r(x)\,dx = \frac{1}{\theta_2}\left[1+\alpha\left(\frac{n-2r+1}{n+1}\right)\left(1-\frac{2y}{\theta_2}\right)\right];\ 0<y<\theta_2,$$

where $f_r(x)$ is the density function of $X_{(r)r}$, i.e.,

$$f_r(x) = \frac{n!}{(r-1)!(n\text{-}r)!}\left[\frac{x^{r-1}(\theta_1-x)^{n-r}}{\theta_1^n}\right];\ 0<x<\theta_1,$$

and therefore, the mean and variance of $Y_{[r]r}$ for $1\le r\le n$ are, respectively, given by

$$E\left[Y_{[r]r}\right]=\theta_2\beta_r \quad \text{and} \quad \mathrm{Var}\left[Y_{[r]r}\right]=\theta_2^2\lambda_r, \tag{11.3}$$

where

$$\beta_r = \frac{1}{2}\left[1-\frac{\alpha}{3}\left(\frac{n-2r+1}{n+1}\right)\right] \quad \text{and} \quad \lambda_r = \frac{1}{12}\left[1-\frac{\alpha^2}{3}\left(\frac{n-2r+1}{n+1}\right)^2\right]$$

Let $Y_{[r]r}, r=1,2,\ldots,n$, be the RSS observations made on the units of the ranked set sampling regarding the study variable Y, which is correlated with the auxiliary variable X, when (X,Y) follows

MTBUD as defined in Eq. (11.2). Then an unbiased estimator for θ_2 based on RSS mean in Eq. (11.3) is given as [see Tahmasebi and Jafari (2012)]

$$t_1 = \hat{\theta}_{2,\mathrm{RSS}} = \frac{2}{n}\sum_{r=1}^{n} Y_{[r]r},$$

and its variance is

$$\mathrm{Var}(t_1) = \frac{\theta_2^2}{3n}\left[1 - \frac{\alpha^2}{3n}\sum_{r=1}^{n}\left(\frac{n-2r+1}{n+1}\right)^2\right] = \theta_2^2 V_1, \tag{11.4}$$

where

$$V_1 = \frac{1}{3n}\left[1 - \frac{\alpha^2}{3n}\sum_{r=1}^{n}\left(\frac{n-2r+1}{n+1}\right)^2\right].$$

When the parameter α is known, Tahmasebi and Jafari (2012) have suggested a BLUE θ_2^* of θ_2, which is more efficient than the estimator $\hat{\theta}_{2,\mathrm{RSS}}$ and is given as:

$$t_2 = \theta_2^* = \sum_{r=1}^{n}\left(\frac{\beta_r}{\lambda_r}\right)\left(\sum_{i=1}^{n}\left(\frac{\beta_i^2}{\lambda_i}\right)\right)^{-1} Y_{[r]r},$$

whose variance is

$$\mathrm{Var}(t_2) = \theta_2^2\left(\sum_{r=1}^{n}\left(\frac{\beta_r^2}{\lambda_r}\right)\right)^{-1} = \theta_2^2 V_2, \tag{11.5}$$

where

$$V_2 = \left(\sum_{r=1}^{n}\left(\frac{\beta_r^2}{\lambda_r}\right)\right)^{-1}.$$

Further, Tahmasebi and Jafari (2012) derived BLUE of θ_2 based on the upper ranked set sample (URSS) as

$$t_3 = \tilde{\theta}_2 = \frac{1}{n\beta_n}\sum_{r=1}^{n} Y_{[n]r},$$

and its variance is given by

$$\mathrm{Var}(t_3) = \theta_2^2\frac{\lambda_n}{n\beta_n^2} = \theta_2^2 V_3, \tag{11.6}$$

where

$$V_3 = \frac{\lambda_n}{n\beta_n^2}.$$

Using the extreme ranked set sampling (ERSS) method, Tahmasebi and Jafari (2012) also derived different estimators for θ_2 with concomitant variable for n. Below we have used the same notations ERSS_1, ERSS_2 and ERSS_3 as defined in Tahmasebi and Jafari (2012), pp. 134–135).

If n is even then the estimator of the θ_2 using ERSS_1 is defined as

$$t_4 = \hat{\theta}_{2,\text{ERSS}_1} = \frac{2}{n}\sum_{r=1}^{n/2} Y_{[1]2r-1} + Y_{[n]2r},$$

and its variance is given by

$$\text{Var}(t_4) = \frac{\theta_2^2}{3n}\left[1 - \frac{\alpha^2}{3}\left(\frac{n-1}{n+1}\right)^2\right] = \theta_2^2 V_4, \tag{11.7}$$

where

$$V_4 = \frac{1}{3n}\left[1 - \frac{\alpha^2}{3}\left(\frac{n-1}{n+1}\right)^2\right].$$

If n is odd then the estimators of θ_2 using ERSS_2 and ERSS_3 are obtained as

$$t_5 = \hat{\theta}_{2,\text{ERSS}_2} = \frac{2\left(Y_{[1]1} + Y_{[n]2} + Y_{[1]3} + \ldots + Y_{[n](n-1)} + \dfrac{\left(Y_{[1]n} + Y_{[n]n}\right)}{2}\right)}{n},$$

and

$$t_6 = \hat{\theta}_{2,\text{ERSS}_3} = \frac{2\left(Y_{[1]1} + Y_{[n]2} + Y_{[1]3} + \ldots + Y_{[n](n-1)} + Y_{[\frac{n+1}{2}]n}\right)}{n}.$$

The variances of the estimators t_5 and t_6 are, respectively, given by

$$\text{Var}(t_5) = \frac{\theta_2^2}{3n}\left[1 - \frac{\alpha^2(n-1)^3}{3n(n+1)^2} - \frac{1}{2n} + \frac{\alpha^2(2-n)}{6n(n+2)}\right] = \theta_2^2 V_5, \tag{11.8}$$

$$\text{Var}(t_6) = \frac{\theta_2^2}{3n}\left[1 - \frac{\alpha^2(n-1)^3}{3n(n+1)^2}\right] = \theta_2^2 V_6, \tag{11.9}$$

where

$$V_5 = \frac{1}{3n}\left[1 - \frac{\alpha^2(n-1)^3}{3n(n+1)^2} - \frac{1}{2n} + \frac{\alpha^2(2-n)}{6n(n+2)}\right],$$

and

$$V_6 = \frac{1}{3n}\left[1 - \frac{\alpha^2(n-1)^3}{3n(n+1)^2}\right].$$

Al-Saleh and Ananbeh (2007) proposed the concept of moving extreme ranked set sampling (MERSS) with a concomitant variable for the estimation of the means of the bivariate normal distribution. Now, suppose that the random vector (X, Y) has an MTBUD as defined in Eq. (11.2). An unbiased estimator of θ_2 based on MERSS is given by [see Tahmasebi and Jafari (2012)]

$$t_7 = \hat{\theta}_{2,\text{MERSS}} = \frac{1}{n}\sum_{r=1}^{n}\left(Y_{[1]r} + Y_{[n]r}\right),$$

and its variance is

$$\mathrm{Var}(t_7) = \frac{\theta_2^2}{6n}\left[1 - \frac{\alpha^2}{3n}\left(\frac{n-1}{n+1}\right)^2\right] = \theta_2^2 V_7, \tag{11.10}$$

where

$$V_7 = \frac{1}{6n}\left[1 - \frac{\alpha^2}{3n}\left(\frac{n-1}{n+1}\right)^2\right].$$

11.3 THE SUGGESTED FAMILY OF ESTIMATORS FOR THE SCALE PARAMETER θ_2 BASED ON THE A PRIORI INTERVAL

The arithmetic mean (AM), the geometric mean (GM), and the harmonic mean (HM) are measures of location, which are used for suggesting different classes of shrinkage estimators for scale parameter θ_2. Let the prior information of θ_2 be available in the form of an interval whose end points are θ_{21} and θ_{22}, such that $\theta_{21} < \theta_{22}$. We define the following families of shrinkage estimators $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ of θ_2 as

$$\psi_{\theta_2}^{(i)} = \delta t_j + (1-\delta)AGH(l,k) = \delta\left[t_j - AGH(l,k)\right] + AGH(l,k), \tag{11.11}$$

where $t_j, j = 1, 2, \ldots, 7$ is an unbiased estimator of the parameter θ_2,δ is a scalar such that $0 \le \delta \le 1$, and $AGH(l,k) = (\theta_{21}\theta_{22})^l \left(\frac{\theta_{21}+\theta_{22}}{2}\right)^k$ for $i = 1, 2, 3$ corresponding to (l, k) which should be taken as $(0, 1)$, $\left(\frac{1}{2}, 0\right)$ and $(1, -1)$ in $AGH(l, k)$. It is interesting to note that for different values of i we have formed the following classes of estimators:

i. For $i = 1$ and $(l, k) = (0, 1)$, we get the class of estimators as

$$\psi_{\theta_2}^{(1)} = \delta\left[t_j - AGH(0,1)\right] + AGH(0,1) = \delta\left[t_j - \left(\frac{\theta_{21}+\theta_{22}}{2}\right)\right] + \left(\frac{\theta_{21}+\theta_{22}}{2}\right), \tag{11.12}$$

ii. For $i = 2$ and $(l, k) = \left(\frac{1}{2}, 0\right)$, we obtain the class of estimators as

$$\psi_{\theta_2}^{(2)} = \delta\left[t_j - AGH\left(\frac{1}{2}, 0\right)\right] + AGH\left(\frac{1}{2}, 0\right) = \delta\left[t_j - \sqrt{\theta_{21}\theta_{22}}\right] + \sqrt{\theta_{21}\theta_{22}}, \tag{11.13}$$

iii. For $i = 3$ and $(l, k) = (1, -1)$, we get the class of estimators as

$$\psi_{\theta_2}^{(3)} = \delta\left[t_j - AGH(1,-1)\right] + AGH(1,-1) = \delta\left[t_j - \left(\frac{2\theta_{21}\theta_{22}}{\theta_{21}+\theta_{22}}\right)\right] + \left(\frac{2\theta_{21}\theta_{22}}{\theta_{21}+\theta_{22}}\right). \tag{11.14}$$

The bias and MSE of $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ are, respectively, given by

$$B\left[\psi_{\theta_2}^{(i)}\right] = \theta_2(1-\delta)\left(\lambda_{(i)} - 1\right) \tag{11.15}$$

$$\mathrm{MSE}\left[\psi_{\theta_2}^{(i)}\right] = \theta_2^2\left[V_j\delta^2 + (1-\delta)^2\left(\lambda_{(i)} - 1\right)^2\right], \tag{11.16}$$

where $\lambda_{(i)} = \frac{AGH(l,k)}{\theta_2}$.

The minimum mean squared error (MMSE) estimators of the parameter θ_2 based on $t_j, j = 1, 2, \ldots, 7$ are given as

$$T_j^* = \frac{\theta_2}{(1 + V_j)}, j = 1, 2, \ldots, 7, \tag{11.17}$$

in the class of estimator $T_j = t_j A_j, j = 1, 2, \ldots, 7$, where $A_j^{'} s, j = 1, 2, \ldots, 7$ are suitably chosen constants such that the MSE of $T_j^{'} s, j = 1, 2, \ldots, 7$ are minimum.

The bias and MSE of $T_j^* s, j = 1, 2, \ldots, 7$ are, respectively, given by

$$B\left(T_j^*\right) = -\theta_2\left(\frac{V_j}{1 + V_j}\right), \tag{11.18}$$

$$\mathrm{MSE}\left(T_j^*\right) = \theta_2^2\left(\frac{V_j}{1 + V_j}\right). \tag{11.19}$$

Comparisons of the proposed shrinkage estimators $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ with that of corresponding usual unbiased estimators $t_j^{'} s, j = 1, 2, \ldots, 7$ are given in Theorem 1.1.

Theorem 1.1: *The proposed shrinkage estimators* $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ *are better than the corresponding usual unbiased estimators* $t_j^{'} s, j = 1, 2, \ldots, 7$ *if*

$$\frac{\left\{\left(\lambda_{(i)} - 1\right)^2 - V_j\right\}}{\left\{\left(\lambda_{(i)} - 1\right)^2 + V_j\right\}} < \delta < 1.$$

Proof: *From Eqs. (11.4)–(11.10) and (11.16), we have that*

$$Var\left(t_j\right) - MSE\left[\psi_{\theta_2}^{(i)}\right] > 0, \quad i = 1, 2, 3, \quad j = 1, 2, \ldots, 7 \text{ if}$$

$$\theta_2^2 V_j - \theta_2^2 V_j \delta^2 - (1-\delta)^2\left(\lambda_{(i)} - 1\right)^2 \theta_2^2 > 0,$$

i.e., if $V_j\left(1 - \delta^2\right) > (1-\delta)^2\left(\lambda_{(i)} - 1\right)^2$, *i.e., if* $V_j(1 + \delta) > (1 - \delta)\left(\lambda_{(i)} - 1\right)^2$,

Now

$$(1 - \delta) > 0 \Rightarrow 1 > \delta \Rightarrow \delta < 1 \tag{11.20}$$

and $V_j + \delta\left\{V_j + \left(\lambda_{(i)} - 1\right)^2\right\} > \left(\lambda_{(i)} - 1\right)^2$, *or* $\delta\left\{V_j + \left(\lambda_{(i)} - 1\right)^2\right\} > \left\{\left(\lambda_{(i)} - 1\right)^2 - V_j\right\}$, *i.e., if*

$$\delta > \frac{\left\{\left(\lambda_{(i)} - 1\right)^2 - V_j\right\}}{\left\{\left(\lambda_{(i)} - 1\right)^2 + V_j\right\}}. \tag{11.21}$$

From Eqs. (11.20) and (11.21) we have

$$\frac{\left\{\left(\lambda_{(i)} - 1\right)^2 - V_j\right\}}{\left\{\left(\lambda_{(i)} - 1\right)^2 + V_j\right\}} < \delta < 1. \tag{11.22}$$

Hence the theorem.♦

Comparisons of the proposed shrinkage estimators $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ with that of corresponding MMSE estimators $T_j^* s, j = 1, 2, \ldots, 7$ are given in Theorem 1.2.

Theorem 1.2: *The proposed shrinkage estimators* $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ *are better than the corresponding MMSE estimators* $T_j^* s, j = 1, 2, \ldots, 7$ *if*

$$\left\{\frac{\left(\lambda_{(i)}-1\right)^2}{\left(\lambda_{(i)}-1\right)^2+V_j} - \frac{V_j\sqrt{\left\{1-\left(\lambda_{(i)}-1\right)^2\right\}}}{\sqrt{(1+V_j)\left\{\left(\lambda_{(i)}-1\right)^2+V_j\right\}}}\right\} < \delta < \left\{\frac{\left(\lambda_{(i)}-1\right)^2}{\left(\lambda_{(i)}-1\right)^2+V_j} + \frac{V_j\sqrt{\left\{1-\left(\lambda_{(i)}-1\right)^2\right\}}}{\sqrt{(1+V_j)\left\{\left(\lambda_{(i)}-1\right)^2+V_j\right\}}}\right\} \tag{11.23}$$

Proof: *From Eqs. (11.16) and (11.19), we have that*

$$MSE\left(T_j^*\right) - MSE\left[\psi_{\theta_2}^{(i)}\right] > 0, \quad i = 1, 2, 3, \quad j = 1, 2, \ldots, 7 \text{ if}$$

$$\theta_2^2 \frac{V_j}{1+V_j} - \theta_2^2 V_j \delta^2 - (1-\delta)^2 \left(\lambda_{(i)}-1\right)^2 \theta_2^2 > 0,$$

i.e., if $-\frac{V_j}{1+V_j} + V_j\delta^2 + \left(1+\delta^2-2\delta\right)\left(1-\delta^2\right)\left(\lambda_{(i)}-1\right)^2 < 0,$

i.e., if $\delta^2\left[-V_j + \left(\lambda_{(i)}-1\right)^2\right] - 2\delta\left(\lambda_{(i)}-1\right)^2 - \frac{V_j}{1+V_j} + \left(\lambda_{(i)}-1\right)^2 > 0,$

On solving the above quadratic equation with respect to δ *we have*

$$\left\{\frac{\left(\lambda_{(i)}-1\right)^2}{\left(\lambda_{(i)}-1\right)^2+V_j} - \frac{V_j\sqrt{\left\{1-\left(\lambda_{(i)}-1\right)^2\right\}}}{\sqrt{(1+V_j)\left\{\left(\lambda_{(i)}-1\right)^2+V_j\right\}}}\right\} < \delta < \left\{\frac{\left(\lambda_{(i)}-1\right)^2}{\left(\lambda_{(i)}-1\right)^2+V_j} + \frac{V_j\sqrt{\left\{1-\left(\lambda_{(i)}-1\right)^2\right\}}}{\sqrt{(1+V_j)\left\{\left(\lambda_{(i)}-1\right)^2+V_j\right\}}}\right\}.$$

Hence the theorem.♦

11.4 RELATIVE EFFICIENCY

We note here that among these seven estimators t_j, $j = 1, 2, \ldots, 7$ discussed above, the estimator t_2 is the best as we have observed numerically. Keeping this in view we have made an effort to compare the estimators $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ formulated based on the BLUE with that of the BLUE t_2 and its MMSE estimator T_2^* by using following the formula:

$$e_1^{(i)} = RE\left(\psi_{\theta_2}^{(i)}, t_2\right) = \frac{V_2}{\left\{V_2\delta^2 + (1-\delta)^2\left(\lambda_{(i)}-1\right)^2\right\}}, \quad i = 1, 2, 3, \tag{11.24}$$

$$e_2^{(i)} = RE\left(\psi_{\theta_2}^{(i)}, T_2^*\right) = \frac{V_2}{(1+V_2)\left\{V_2\delta^2 + (1-\delta)^2\left(\lambda_{(i)}-1\right)^2\right\}}, \quad i = 1, 2, 3. \tag{11.25}$$

The values of $e_1^{(i)}$ and $e_2^{(i)}$, $i = 1, 2, 3$ are shown in Table 11.1 for $n = 5(5)20$, $\alpha = 0.25(0.25)1.00$ and different values of $\psi_1 = \frac{\theta_{21}}{\theta_2} = 0.5(0.1)0.9$, $\psi_2 = \frac{\theta_{22}}{\theta_2} = 1.1(0.1)1.5$ and $\delta = 0.25(0.25)0.75$.

Table 11.1 The Values of $e_1^{(i)}$ and $e_2^{(i)'}s$, $i = 1, 2, 3$ for Different Values of n, (ψ_1, ψ_2), δ and Fixed $\alpha = 0.25$

$(\psi_1,\psi_2)\rightarrow$ $n\downarrow$	δ	(0.5,1.1)			(0.6,1.2)			(0.7,1.3)			(0.8,1.4)			(0.9,1.5)		
		$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$
5	0.25	2.4869	1.5882	1.1211	6.7842	3.8922	2.4869	16.0000	12.4488	7.6179	6.7842	10.9450	15.3735	2.4869	3.5095	5.1297
	0.50	2.4942	1.9918	1.6164	3.4754	2.9726	2.4942	4.0000	3.8771	3.5642	3.4754	3.8047	3.9820	2.4942	2.8665	3.2377
	0.75	1.6660	1.5987	1.5275	1.7485	1.7120	1.6660	1.7778	1.7715	1.7540	1.7485	1.7677	1.7769	1.6660	1.7030	1.7325
10	0.25	1.3465	0.8344	0.5801	4.3002	2.2128	1.3465	16.0000	10.1822	4.9940	4.3002	8.3114	14.7926	1.3465	1.9684	3.0509
	0.50	1.8106	1.3248	1.0117	3.0715	2.3637	1.8106	4.0000	3.7612	3.2132	3.0715	3.6272	3.9640	1.8106	2.2321	2.7181
	0.75	1.5672	1.4520	1.3385	1.7200	1.6508	1.5672	1.7778	1.7653	1.7307	1.7200	1.7577	1.7760	1.5672	1.6340	1.6893
15	0.25	0.9231	0.5658	0.3912	3.1475	1.5458	0.9231	16.0000	8.6137	3.7144	3.1475	6.6992	14.2539	0.9231	1.3677	2.1709
	0.50	1.4210	0.9924	0.7363	2.7516	1.9618	1.4210	4.0000	3.6520	2.9250	2.7516	3.4654	3.9463	1.4210	1.8276	2.3422
	0.75	1.4794	1.3299	1.1911	1.6925	1.5938	1.4794	1.7778	1.7592	1.7080	1.6925	1.7478	1.7751	1.4794	1.5704	1.6482
20	0.25	0.7023	0.4281	0.2952	2.4822	1.1878	0.7023	16.0000	7.4640	2.9569	2.4822	5.6110	13.7531	0.7023	1.0479	1.6850
	0.50	1.1695	0.7933	0.5787	2.4921	1.6767	1.1695	4.0000	3.5490	2.6843	2.4921	3.3175	3.9287	1.1695	1.5472	2.0577
	0.75	1.4010	1.2268	1.0730	1.6658	1.5406	1.4010	1.7778	1.7530	1.6860	1.6658	1.7380	1.7742	1.4010	1.5115	1.6090

$(\psi_1,\psi_2)\rightarrow$ $n\downarrow$	δ	(0.5,1.1)			(0.6,1.2)			(0.7,1.3)			(0.8,1.4)			(0.9,1.5)		
		$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$
5	0.25	2.3324	1.4895	1.0514	6.3627	3.6504	2.3324	15.0058	11.6753	7.1446	6.3627	10.2649	14.4182	2.3324	3.2915	4.8109
	0.50	2.3392	1.8680	1.5160	3.2595	2.7879	2.3392	3.7515	3.6362	3.3428	3.2595	3.5683	3.7345	2.3392	2.6884	3.0365
	0.75	1.5625	1.4993	1.4326	1.6398	1.6056	1.5625	1.6673	1.6615	1.6450	1.6398	1.6579	1.6665	1.5625	1.5971	1.6248
10	0.25	1.3033	0.8077	0.5615	4.1625	2.1420	1.3033	15.4877	9.8562	4.8341	4.1625	8.0453	14.3189	1.3033	1.9054	2.9532
	0.50	1.7526	1.2823	0.9794	2.9731	2.2880	1.7526	3.8719	3.6408	3.1103	2.9731	3.5110	3.8371	1.7526	2.1606	2.6311
	0.75	1.5170	1.4055	1.2957	1.6649	1.5979	1.5170	1.7209	1.7088	1.6753	1.6649	1.7014	1.7191	1.5170	1.5817	1.6352
15	0.25	0.9032	0.5536	0.3828	3.0796	1.5124	0.9032	15.6550	8.4279	3.6343	3.0796	6.5547	13.9465	0.9032	1.3382	2.1241
	0.50	1.3904	0.9710	0.7204	2.6922	1.9194	1.3904	3.9137	3.5733	2.8620	2.6922	3.3907	3.8612	1.3904	1.7881	2.2917
	0.75	1.4475	1.3012	1.1654	1.6560	1.5594	1.4475	1.7394	1.7212	1.6712	1.6560	1.7101	1.7368	1.4475	1.5365	1.6126
20	0.25	0.6909	0.4211	0.2904	2.4419	1.1684	0.6909	15.7399	7.3427	2.9088	2.4419	5.5197	13.5295	0.6909	1.0309	1.6576
	0.50	1.1505	0.7804	0.5693	2.4516	1.6494	1.1505	3.9350	3.4913	2.6407	2.4516	3.2636	3.8648	1.1505	1.5220	2.0242
	0.75	1.3782	1.2069	1.0555	1.6387	1.5155	1.3782	1.7489	1.7245	1.6586	1.6387	1.7098	1.7454	1.3782	1.4870	1.5829

(For Fixed $\alpha = 0.50$)

$(\psi_1,\psi_2)\rightarrow$ $n\downarrow$	δ	(0.5, 1.1)			(0.6, 1.2)			(0.7, 1.3)			(0.8, 1.4)			(0.9, 1.5)		
		$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$
5	0.25	2.4473	1.5612	1.1014	6.7101	3.8365	2.4473	16.0000	12.3960	7.5421	6.7101	10.8790	15.3620	2.4473	3.4577	5.0636
	0.50	2.4763	1.9728	1.5981	3.4667	2.9580	2.4763	4.0000	3.8748	3.5568	3.4667	3.8012	3.9816	2.4763	2.8510	3.2259
	0.75	1.6640	1.5956	1.5234	1.7479	1.7108	1.6640	1.7778	1.7714	1.7535	1.7479	1.7675	1.7769	1.6640	1.7016	1.7316

Table 11.1 The Values of $e_1^{(i)}$ and $e_2^{(i)'}s$, $i = 1, 2, 3$ for Different Values of n, (ψ_1, ψ_2), δ and Fixed $\alpha = 0.25$ *Continued*

(For Fixed $\alpha = 0.50$)

$(\psi_1,\psi_2)\rightarrow$ $n\downarrow$	δ	(0.5, 1.1)			(0.6, 1.2)			(0.7, 1.3)			(0.8, 1.4)			(0.9, 1.5)		
		$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$
10	0.25	1.3178	0.8160	0.5671	4.2267	2.1684	1.3178	16.0000	10.0950	4.9137	4.2267	8.2176	14.7661	1.3178	1.9282	2.9933
	0.50	1.7873	1.3040	0.9941	3.0546	2.3409	1.7873	4.0000	3.7559	3.1982	3.0546	3.6192	3.9632	1.7873	2.2089	2.6976
	0.75	1.5628	1.4457	1.3307	1.7187	1.6480	1.5628	1.7778	1.7650	1.7296	1.7187	1.7572	1.7759	1.5628	1.6308	1.6873
15	0.25	0.9014	0.5523	0.3818	3.0844	1.5110	0.9014	16.0000	8.5136	3.6432	3.0844	6.6014	14.2144	0.9014	1.3366	2.1242
	0.50	1.3981	0.9737	0.7213	2.7299	1.9366	1.3981	4.0000	3.6440	2.9051	2.7299	3.4537	3.9449	1.3981	1.8026	2.3177
	0.75	1.4732	1.3214	1.1812	1.6904	1.5896	1.4732	1.7778	1.7587	1.7063	1.6904	1.7471	1.7750	1.4732	1.5657	1.6451
20	0.25	0.6850	0.4173	0.2877	2.4281	1.1594	0.6850	16.0000	7.3603	2.8946	2.4281	5.5163	13.7023	0.6850	1.0227	1.6461
	0.50	1.1480	0.7769	0.5659	2.4675	1.6514	1.1480	4.0000	3.5385	2.6612	2.4675	3.3026	3.9268	1.1480	1.5225	2.0316
	0.75	1.3932	1.2168	1.0619	1.6630	1.5352	1.3932	1.7778	1.7524	1.6837	1.6630	1.7370	1.7741	1.3932	1.5056	1.6050

$(\psi_1,\psi_2)\rightarrow$ $n\downarrow$	δ	(0.5, 1.1)			(0.6, 1.2)			(0.7, 1.3)			(0.8, 1.4)			(0.9, 1.5)		
		$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$
5	0.25	2.2979	1.4659	1.0342	6.3005	3.6023	2.2979	15.0234	11.6394	7.0817	6.3005	10.2150	14.4243	2.2979	3.2467	4.7546
	0.50	2.3251	1.8524	1.5006	3.2551	2.7774	2.3251	3.7558	3.6383	3.3397	3.2551	3.5692	3.7386	2.3251	2.6769	3.0290
	0.75	1.5624	1.4982	1.4304	1.6412	1.6064	1.5624	1.6693	1.6633	1.6465	1.6412	1.6596	1.6684	1.5624	1.5977	1.6259
10	0.25	1.2765	0.7905	0.5494	4.0944	2.1006	1.2765	15.4992	9.7791	4.7599	4.0944	7.9604	14.3039	1.2765	1.8679	2.8996
	0.50	1.7314	1.2632	0.9630	2.9590	2.2676	1.7314	3.8748	3.6383	3.0981	2.9590	3.5059	3.8392	1.7314	2.1397	2.6132
	0.75	1.5139	1.4004	1.2891	1.6649	1.5964	1.5139	1.7221	1.7098	1.6755	1.6649	1.7022	1.7204	1.5139	1.5798	1.6345
15	0.25	0.8825	0.5406	0.3737	3.0195	1.4792	0.8825	15.6633	8.3344	3.5665	3.0195	6.4625	13.9153	0.8825	1.3084	2.0795
	0.50	1.3687	0.9532	0.7061	2.6724	1.8959	1.3687	3.9158	3.5673	2.8440	2.6724	3.3810	3.8619	1.3687	1.7647	2.2690
	0.75	1.4422	1.2936	1.1563	1.6548	1.5561	1.4422	1.7404	1.7217	1.6704	1.6548	1.7103	1.7377	1.4422	1.5328	1.6105
20	0.25	0.6741	0.4107	0.2832	2.3896	1.1410	0.6741	15.7465	7.2437	2.8487	2.3896	5.4289	13.4851	0.6741	1.0065	1.6200
	0.50	1.1298	0.7646	0.5570	2.4284	1.6252	1.1298	3.9366	3.4824	2.6191	2.4284	3.2503	3.8646	1.1298	1.4984	1.9994
	0.75	1.3711	1.1976	1.0450	1.6367	1.5109	1.3711	1.7496	1.7246	1.6570	1.6367	1.7095	1.7460	1.3711	1.4817	1.5796

(For Fixed $\alpha = 0.75$)

$(\psi_1,\psi_2)\rightarrow$ $n\downarrow$	δ	(0.5, 1.1)			(0.6, 1.2)			(0.7, 1.3)			(0.8, 1.4)			(0.9, 1.5)		
		$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$
5	0.25	2.3798	1.5155	1.0682	6.5822	3.7413	2.3798	16.0000	12.3033	7.4110	6.5822	10.7637	15.3415	2.3798	3.3693	4.9504
	0.50	2.4451	1.9399	1.5667	3.4513	2.9324	2.4451	4.0000	3.8708	3.5437	3.4513	3.7949	3.9810	2.4451	2.8238	3.2051
	0.75	1.6605	1.5901	1.5161	1.7469	1.7087	1.6605	1.7778	1.7712	1.7527	1.7469	1.7672	1.7768	1.6605	1.6991	1.7301
10	0.25	1.2687	0.7846	0.5449	4.0995	2.0921	1.2687	16.0000	9.9403	4.7742	4.0995	8.0524	14.7182	1.2687	1.8592	2.8941
	0.50	1.7466	1.2680	0.9636	3.0245	2.3007	1.7466	4.0000	3.7462	3.1714	3.0245	3.6047	3.9617	1.7466	2.1679	2.6610
	0.75	1.5549	1.4344	1.3167	1.7163	1.6429	1.5549	1.7778	1.7645	1.7276	1.7163	1.7564	1.7759	1.5549	1.6252	1.6836

(*Continued*)

Table 11.1 The Values of $e_1^{(i)}$ and $e_2^{(i)'}s$, $i = 1, 2, 3$ for Different Values of n, (ψ_1, ψ_2), δ and Fixed $\alpha = 0.25$ *Continued*

(For Fixed $\alpha = 0.75$)

$(\psi_1,\psi_2)\rightarrow$		(0.5, 1.1)			(0.6, 1.2)			(0.7, 1.3)			(0.8, 1.4)			(0.9, 1.5)		
$n\downarrow$	δ	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$
15	0.25	0.8643	0.5290	0.3655	2.9750	1.4511	0.8643	16.0000	8.3359	3.5193	2.9750	6.4294	14.1425	0.8643	1.2830	2.0434
	0.50	1.3578	0.9413	0.6953	2.6910	1.8921	1.3578	4.0000	3.6292	2.8694	2.6910	3.4323	3.9425	1.3578	1.7586	2.2742
	0.75	1.4617	1.3062	1.1634	1.6866	1.5820	1.4617	1.7778	1.7578	1.7032	1.6866	1.7457	1.7749	1.4617	1.5572	1.6395
20	0.25	0.6553	0.3989	0.2749	2.3342	1.1106	0.6553	16.0000	7.1766	2.7864	2.3342	5.3502	13.6097	0.6553	0.9793	1.5790
	0.50	1.1105	0.7483	0.5438	2.4235	1.6066	1.1105	4.0000	3.5192	2.6197	2.4235	3.2755	3.9234	1.1105	1.4791	1.9853
	0.75	1.3791	1.1989	1.0420	1.6579	1.5253	1.3791	1.7778	1.7512	1.6795	1.6579	1.7351	1.7739	1.3791	1.4947	1.5976

$(\psi_1,\psi_2)\rightarrow$		(0.5, 1.1)			(0.6, 1.2)			(0.7, 1.3)			(0.8, 1.4)			(0.9, 1.5)		
$n\downarrow$	δ	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$
5	0.25	2.2390	1.4258	1.0050	6.1927	3.5199	2.2390	15.0531	11.5752	6.9724	6.1927	10.1267	14.4336	2.2390	3.1699	4.6574
	0.50	2.3004	1.8251	1.4739	3.2471	2.7589	2.3004	3.7633	3.6417	3.3340	3.2471	3.5703	3.7454	2.3004	2.6567	3.0154
	0.75	1.5622	1.4960	1.4264	1.6435	1.6075	1.5622	1.6726	1.6664	1.6490	1.6435	1.6626	1.6717	1.5622	1.5986	1.6277
10	0.25	1.2305	0.7610	0.5285	3.9762	2.0292	1.2305	15.5189	9.6414	4.6306	3.9762	7.8102	14.2756	1.2305	1.8033	2.8070
	0.50	1.6941	1.2299	0.9346	2.9335	2.2315	1.6941	3.8797	3.6336	3.0761	2.9335	3.4963	3.8425	1.6941	2.1027	2.5810
	0.75	1.5081	1.3913	1.2771	1.6647	1.5935	1.5081	1.7243	1.7114	1.6757	1.6647	1.7036	1.7225	1.5081	1.5763	1.6330
15	0.25	0.8469	0.5183	0.3581	2.9151	1.4219	0.8469	15.6777	8.1680	3.4485	2.9151	6.2999	13.8576	0.8469	1.2572	2.0023
	0.50	1.3305	0.9223	0.6813	2.6368	1.8540	1.3305	3.9194	3.5561	2.8116	2.6368	3.3632	3.8631	1.3305	1.7232	2.2284
	0.75	1.4323	1.2799	1.1400	1.6526	1.5501	1.4323	1.7420	1.7224	1.6689	1.6526	1.7105	1.7391	1.4323	1.5259	1.6065
20	0.25	0.6453	0.3929	0.2708	2.2989	1.0938	0.6453	15.7578	7.0680	2.7442	2.2989	5.2692	13.4036	0.6453	0.9644	1.5551
	0.50	1.0937	0.7370	0.5356	2.3868	1.5823	1.0937	3.9394	3.4660	2.5800	2.3868	3.2260	3.8640	1.0937	1.4567	1.9553
	0.75	1.3582	1.1808	1.0262	1.6328	1.5022	1.3582	1.7509	1.7247	1.6540	1.6328	1.7089	1.7471	1.3582	1.4721	1.5734

(For Fixed $\alpha = 1.00$)

$(\psi_1,\psi_2)\rightarrow$		(0.5, 1.1)			(0.6, 1.2)			(0.7, 1.3)			(0.8, 1.4)			(0.9, 1.5)		
$n\downarrow$	δ	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$
5	0.25	2.2820	1.4494	1.0202	6.3926	3.6023	2.2820	16.0000	12.1617	7.2159	6.3926	10.5892	15.3097	2.2820	3.2405	4.7840
	0.50	2.3982	1.8908	1.5201	3.4276	2.8935	2.3982	4.0000	3.8645	3.5234	3.4276	3.7851	3.9801	2.3982	2.7826	3.1733
	0.75	1.6550	1.5817	1.5050	1.7454	1.7053	1.6550	1.7778	1.7709	1.7515	1.7454	1.7666	1.7768	1.6550	1.6954	1.7278
10	0.25	1.1966	0.7387	0.5125	3.9092	1.9795	1.1966	16.0000	9.6998	4.5645	3.9092	7.7988	14.6414	1.1966	1.7576	2.7468
	0.50	1.6845	1.2137	0.9179	2.9769	2.2384	1.6845	4.0000	3.7308	3.1290	2.9769	3.5815	3.9592	1.6845	2.1048	2.6040
	0.75	1.5422	1.4165	1.2948	1.7124	1.6348	1.5422	1.7778	1.7636	1.7244	1.7124	1.7550	1.7757	1.5422	1.6161	1.6778

Table 11.1 The Values of $e_1^{(i)}$ and $e_2^{(i)'}s$, $i = 1, 2, 3$ for Different Values of n, (ψ_1, ψ_2), δ and Fixed $\alpha = 0.25$ *Continued*

(For Fixed $\alpha = 1.00$)

$(\psi_1,\psi_2)\rightarrow$ $n\downarrow$	δ	(0.5, 1.1)			(0.6, 1.2)			(0.7, 1.3)			(0.8, 1.4)			(0.9, 1.5)		
		$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$	$e_1^{(1)}$	$e_1^{(2)}$	$e_1^{(3)}$
15	0.25	0.8096	0.4948	0.3417	2.8115	1.3626	0.8096	16.0000	8.0600	3.3336	2.8115	6.1659	14.0261	0.8096	1.2039	1.9236
	0.50	1.2967	0.8925	0.6566	2.6295	1.8235	1.2967	4.0000	3.6054	2.8126	2.6295	3.3979	3.9384	1.2967	1.6909	2.2062
	0.75	1.4434	1.2819	1.1354	1.6805	1.5696	1.4434	1.7778	1.7564	1.6981	1.6805	1.7434	1.7747	1.4434	1.5436	1.6305
20	0.25	0.6115	0.3719	0.2562	2.1945	1.0385	0.6115	16.0000	6.8933	2.6248	2.1945	5.0973	13.4598	0.6115	0.9152	1.4796
	0.50	1.0538	0.7056	0.5109	2.3543	1.5381	1.0538	4.0000	3.4880	2.5540	2.3543	3.2319	3.9178	1.0538	1.4128	1.9135
	0.75	1.3564	1.1705	1.0108	1.6497	1.5093	1.3564	1.7778	1.7492	1.6726	1.6497	1.7320	1.7736	1.3564	1.4772	1.5857

$(\psi_1,\psi_2)\rightarrow$ $n\downarrow$	δ	(0.5, 1.1)			(0.6, 1.2)			(0.7, 1.3)			(0.8, 1.4)			(0.9, 1.5)		
		$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$	$e_2^{(1)}$	$e_2^{(2)}$	$e_2^{(3)}$
5	0.25	2.1530	1.3675	0.9626	6.0315	3.3988	2.1530	15.0960	11.4746	6.8082	6.0315	9.9909	14.4447	2.1530	3.0574	4.5137
	0.50	2.2627	1.7840	1.4342	3.2340	2.7300	2.2627	3.7740	3.6461	3.3243	3.2340	3.5712	3.7552	2.2627	2.6254	2.9940
	0.75	1.5614	1.4924	1.4199	1.6468	1.6090	1.5614	1.6773	1.6708	1.6525	1.6468	1.6668	1.6764	1.5614	1.5996	1.6301
10	0.25	1.1627	0.7178	0.4980	3.7986	1.9235	1.1627	15.5476	9.4255	4.4355	3.7986	7.5783	14.2274	1.1627	1.7079	2.6691
	0.50	1.6368	1.1794	0.8920	2.8928	2.1751	1.6368	3.8869	3.6253	3.0405	2.8928	3.4802	3.8472	1.6368	2.0453	2.5304
	0.75	1.4986	1.3764	1.2581	1.6640	1.5886	1.4986	1.7275	1.7138	1.6757	1.6640	1.7054	1.7255	1.4986	1.5704	1.6304
15	0.25	0.7943	0.4855	0.3352	2.7586	1.3369	0.7943	15.6988	7.9083	3.2708	2.7586	6.0498	13.7621	0.7943	1.1812	1.8874
	0.50	1.2722	0.8757	0.6442	2.5800	1.7892	1.2722	3.9247	3.5375	2.7596	2.5800	3.3339	3.8643	1.2722	1.6591	2.1646
	0.75	1.4162	1.2577	1.1140	1.6488	1.5401	1.4162	1.7443	1.7234	1.6662	1.6488	1.7106	1.7413	1.4162	1.5145	1.5998
20	0.25	0.6029	0.3666	0.2525	2.1636	1.0239	0.6029	15.7743	6.7961	2.5878	2.1636	5.0254	13.2699	0.6029	0.9023	1.4587
	0.50	1.0389	0.6956	0.5037	2.3211	1.5164	1.0389	3.9436	3.4388	2.5180	2.3211	3.1863	3.8626	1.0389	1.3928	1.8865
	0.75	1.3373	1.1540	0.9965	1.6264	1.4880	1.3373	1.7527	1.7246	1.6490	1.6264	1.7076	1.7486	1.3373	1.4564	1.5633

11.5 CONCLUSION

It is observed from Table 11.1 that:

- when $(\psi_1, \psi_2) \in (0.7, 1.3)$ the proposed classes of estimators $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ is always better than the usual unbiased estimator t_2 and MMSE estimator T_2^*;
- the gain in efficiency by using $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ over MMSE estimator T_2^* is fewer than by using $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$ over the BLUE t_2;
- for $(\psi_1, \psi_2) \in (0.7, 1.3)$, the developed class of estimators $\psi_{\theta_2}^{(1)}$ (based on AM) is the best (best in the sense of having smaller MSE) among $\psi_{\theta_2}^{(i)}(i = 1, 2, 3)$, while for $(\psi_1, \psi_2) \in (0.9, 1.5)$ the developed class of estimator $\psi_{\theta_2}^{(3)}$ (based on HM) is the best among $\psi_{\theta_2}^{(i)}$ $(i = 1, 2, 3)$.

In general the proposed estimator $\psi_{\theta_2}^{(1)}$ is recommended when $(\psi_1, \psi_2) \in (0.5, 1.3)$ and $\psi_{\theta_2}^{(3)}$ is recommended when $(\psi_1, \psi_2) \in (0.8, 1.5)$ and the sample size n is small. In practice, when the observations are expensive such small sizes may be all that are available, particularly in defense weapon testing problems.

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