SOME NEW APPROACHES TO SOLVE DECISION MAKING PROBLEMS UNDER PYTHAGOREAN FUZZY SET ENVIRONMENT

Thesis submitted in fulfillment of the requirements for the Degree of

DOCTOR OF PHILOSOPHY

By

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DECLARATION BY THE SCHOLAR

I hereby declare that the work reported in the Ph.D. thesis entitled "**Some New Approaches to Solve Decision Making Problems Under Pythagorean Fuzzy Set Environment**" at **Jaypee University of Information Technology, Waknaghat, Solan (H.P.) India**, is an authentic record of my work carried out under the supervision of **Dr. Rakesh Kumar Bajaj**. I have not submitted this work elsewhere for any other degree or diploma. I am fully responsible for the contents of my Ph.D. Thesis.

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SUPERVISOR'S CERTIFICATE

This is to certify that the work reported in the Ph.D. thesis entitled "**Some New Approaches to Solve Decision Making Problems Under Pythagorean Fuzzy Set Environment**" submitted by **Abhishek Guleria** at **Jaypee University of Information Technology, Waknaghat,** is a bonafide record of his original work carried out under my supervision. This work has not been submitted elsewhere for any other degree or diploma.

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(Abhishek Guleria)

Abstract

The objective of this thesis entitled, "*Some New Approaches to Solve Decision Making Problems Under Pythagorean Fuzzy Set Environment*", is to study new notions of Pythagorean Fuzzy soft matrices and information measures with their applications in decision making processes. The way we think, we process information, we make our decision and particularly in our language, fuzziness can be found everywhere. In real world scenario, decision making is the biggest challenge now a days due to its significance and importance everywhere such as in companies, in industries, in institutions and many more. Thus it is the need of the hour to handle uncertainty, vagueness and impreciseness involved in the decision making problems by developing some new techniques.

The reported work in the thesis is classified into two categories: one is related to the notion of soft matrices and other is related to the information measures. The main goal of the thesis is to deal with the uncertainty, vagueness and impreciseness available in the informational data and solve the multi-criteria decision making problems. For handling such circumstances, we have utilized the several extensions of fuzzy set theory as Pythagorean fuzzy set. The work related to the thesis is described as:

In Chapter 1, we have presented the preliminaries related to the proposed work, which covers all the basic definitions related to the extensions of fuzzy set theory to Pythagorean fuzzy set along with the literature reviewed on the soft matrices and information measure such as entropy, divergence.

In Chapter 2, we have developed the new kind of soft matrix called Pythagorean fuzzy soft matrix with its different possible types and also presented binary operations satisfying various properties with the proof of their validity. Some new kinds of matrices such as choice matrix, weighted choice/score matrix, & utility matrix have also been proposed in a modified format. Further, we have utilized these matrices to solve the multi-criteria decision making problem, medical diagnosis problem and presented some observed comparative remarks in contrast with the other existing methods.

The dimensionality reduction plays an effective role in downsizing the data having irregular factors and acquires an arrangement of important factors in the information. Sometimes, most of the attributes in the information are found to be correlated and hence redundant. The process of dimensionality reduction has a wider applicability in dealing with the decision making problems where a large number of factors are involved. In Chapter 3, we have presented an algorithm for the dimensional reduction of the informational data under Pythagorean fuzzy setup by using the proposed definitions of the

object-oriented matrix, the parameter-oriented Pythagorean fuzzy soft matrix and the threshold value. We have illustrated the methodology of the proposed technique to solve the multi-criteria decision making problem and also provided the comparative remarks & additional advantages of the technique in view of some existing recent methodologies.

In Chapter 4 & 5, we have developed a parametric entropy measure and also a divergence measure for the Pythagorean fuzzy set along with their proof of validity respectively. The monotonic property of these information measures in relation with their parameters have also been studied and presented in these chapters. Further, we have implemented these measures in providing different algorithms for solving multi-criteria decision making and other soft computing applications. The comparative analysis has also been presented for clearly depicting the important observations and advantages of the proposed methodologies in these chapters.

In Chapter 6, we proposed the modified VIKOR and modified TOPSIS multi-criteria decision making technique by incorporating (R, S)-Norm Pythagorean fuzzy entropy and respective discriminant measure in two different stages. Further, the proposed techniques have been implemented and illustrated by solving the hydrogen power plant site selection problem with proper matching of the laid down essential criteria under a wider sense of Pythagorean fuzzy information. A detailed comparative analysis and the sensitivity analysis have been carried out for a better understanding and clarity of the proposed methodologies. Finally, the works reported in this thesis have been concluded in Chapter 7.

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Journal Articles

- A. Guleria, R. K. Bajaj, ***On Pythagorean fuzzy soft matrices, operations and their applications in decision making and medical diagnosis***, *Soft Computing*, vol. 23, pp. 7889–7900, 2019. (**Indexing:** SCI, SCOPUS, IF- 3.050)
- R. K. Bajaj, A. Guleria, ***Dimensionality Reduction Technique in Decision Making Using Pythagorean Fuzzy Soft Matrices***, *Recent Advances in Computer Science and Communications (Formerly: Recent Patents on Computer Science)*, vol. 13(3), pp. 406-413, 2020. (**Indexing:** SCOPUS)
- A. Guleria, R. K. Bajaj, **Pythagorean Fuzzy (R, S)-Norm Information Measure for Multicriteria Decision-Making Problem**, *Advances in Fuzzy Systems*, vol. 2018, pp. 1-11, 2018. (**Indexing:** ESCI, SCOPUS)
- A. Guleria, R. K. Bajaj., **Pythagorean Fuzzy (R, S)-norm Discriminant Measure in Various Decision Making Processes**, *Journal of Intelligent & Fuzzy Systems*, vol. 38, pp. 761-777, 2020. (**Indexing:** SCI, SCOPUS, IF- 1.815)
- A. Guleria, R. K. Bajaj, **A Robust Decision Making Approach for Hydrogen Power Plant Site Selection Utilizing (R, S)-Norm Pythagorean Fuzzy Information Measures Based on VIKOR and TOPSIS Method**, *International Journal of Hydrogen Energy*, vol. 45(38), pp. 18802-18816, 2020. (**Indexing:** SCI, SCOPUS, IF- 4.939)

Conference Paper Presented

- **"On Bi-parametric Pythagorean Fuzzy Information Measure with Applications"** *in Interdisciplinary Approaches in Understanding Science, Arts and Spirituality*, jointly organized by Pratibha Spandan Society, Shimla and Department of Visual Arts, Himachal Pradesh University, Shimla, held on 1^{st} – 2^{nd}, October, 2018, at HPU, Summer Hills, Shimla, Himachal Pradesh.
- **"Computational Application of Bi-parametric (R, S)-Norm Pythagorean Fuzzy Discriminant Measure"** *in Recent Trends in Interdisciplinary Research*, jointly organized by Department of Visual Arts and Pratibha Spandan Society,

Shimla , Himachal Pradesh University, Shimla, held on 9^{th} – 10^{th}, September, 2019, at HPU, Summer Hills, Shimla, H.P., INDIA.

- **"Modified VIKOR Decision Making Algorithm Based on (R, S)-Norm Pythagorean Fuzzy Information Measures"** *in Recent Advances in Mathematical Sciences and its Applications (RAMSA-2020)*, organized by the Department of Mathematics at Jaypee Institute of Information Technology, Noida, Uttar Pradesh, India, held on $9^{th} - 11^{th}$, Jan, 2020.

Chapter 1

Introduction

Decision making is a very significant area and its omnipresence in business, manufacturing, services etc. are quite enough to understand its importance. It is a cognitive process/method to choose the best/optimal alternative among the available alternatives. There are several decisions in our daily routine which have the immediate or long-term effect on us or others whether it may be related to our personal life or professional life. As far as the significance of decision-making is concerned, we all know that the survival of people, growth in business, promotion in jobs etc. are totally dependent on the potential of decision making tasks. The process to select the best/optimal option among available ones with multiple, usually conflicting criteria under the presence of one/many decision makers is applied in multi-criteria decision making problems.

In different practical and real-life situations, the way towards making a decision is strongly roused by the advantages out of it and it also depends on our perception and prior information. In view of the deficiency in the information and possibility of human errors, it is probably expected to have inherited complexity in the environment and having incomplete knowledge of the systems. In this way, it appears to be tough to get an optimal decision in a stipulated time. As the complexities are increasing day by day, decision makers come across many problems to decide within a reasonable time by using the information which is vague/uncertain/imprecise in nature. Pythagorean fuzzy set theory [106], an extension of novel concept of Zadeh's fuzzy set [68] is one of the most acceptable theory to deal with the uncertainties, vagueness and incompleteness in the information.

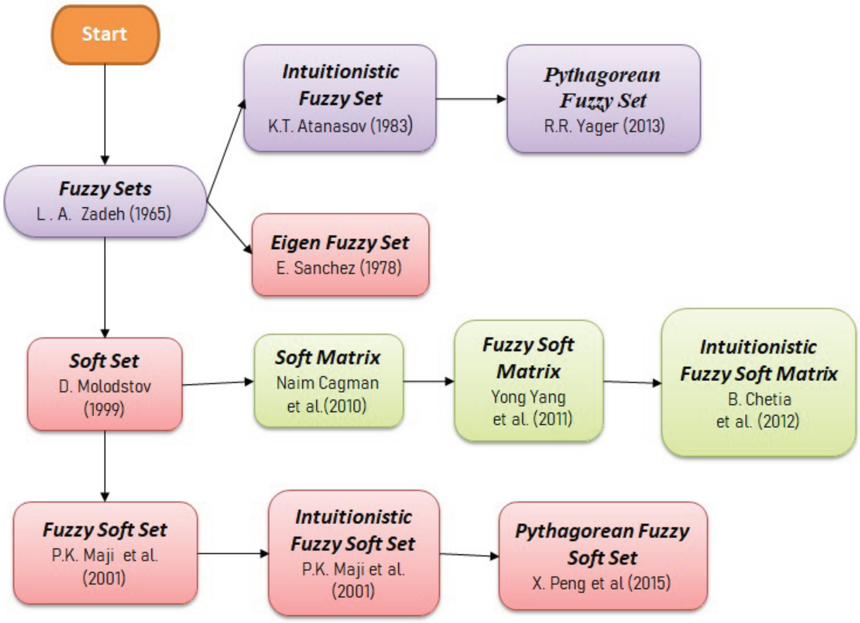

Figure 1.1: Extensions of Fuzzy Set

1.1 Basic Notions and Preliminaries

In this section, we have presented some basic notions related to the Pythagorean fuzzy set.

1.1.1 Fuzzy Set

The crisp or classical set is a well defined collection of elements/objects from the universe of discourse or universal set ($u \in U$), that can be finite/countable/uncountable. Every individual member of the crisp set $A \subset U$, either belong to A ($u \in A$) or does not belong to A ($u \notin A$). A classical/crisp set can be represented in two ways: one can either present the set analytically or enumerate the elements which belong to the set.

An ordinary or *crisp* set A in a universal set U can be described by listing all its members or by defining the conditions to classify the elements $u \in A$, i.e., $A = \{u|u \text{ meets some condition}\}$. The characteristic function χ associated with A, is a mapping $\chi_A : U \to \{0, 1\}$ such that for any element $u \in U$, $\chi_A(u) = 1$, if $u \in A$ and $\chi_A(u) = 0$, if $u \notin A$.

Fuzzy set (FS) [68] is an extension of crisp set. Any fuzzy set A over a set U (universe of discourse) can be characterized by its membership function, i.e., $\mu_A : U \to [0, 1]$ and the output value given by this function represents the grade of degree to which an element of the set U belongs to the set A. Thus, an element in the fuzzy set may belong to a greater or lesser degree as represented by a larger or smaller membership grade.

Remark: Fuzziness is often confused with probability. An event is probabilistic if it has a degree of actual occurrence or it has the results of well identified but random occurrence, i.e., probability measures the likelihood of a future event based on something known now. On the other hand, fuzziness depicts the lack of distinction of an event, whereas the probability describes the uncertainty in the occurrence of the event. In other words, probability relates to randomness and is not an efficient concept to counter the issue of uncertainty and impreciseness resulting due to incompleteness in the informaton.

Definition 1.1.1 *[68] "Consider two fuzzy sets A and B over the universe of discourse set U. The binary operations defined over the fuzzy sets are as:*

- ***Intersection:*** $\mu_{A\cap B}(u) = \min\{\mu_A(u), \mu_B(u)\}, u \in U$.
- ***Union:*** $\mu_{A\cup B}(u) = \max\{\mu_A(u), \mu_B(u)\}, u \in U$.
- ***Complement:*** $\mu_{\bar{A}}(u) = 1 - \mu_A(u), u \in U$.
- ***Probabilistic Sum:*** $\mu_{A+B}(u) = \mu_A(u) + \mu_B(u) - \mu_A(u) \cdot \mu_B(u), u \in U$.
- ***Bounded Sum:*** $\mu_{A\oplus B}(u) = \min\{1, \mu_A(u) + \mu_B(u)\}, u \in U$.
- ***Bounded Difference:*** $\mu_{A\ominus B}(u) = \max\{0, \mu_A(u) + \mu_B(u) - 1\}, u \in U$.
- ***Algebraic Product:*** $\mu_{A\cdot B}(u) = \mu_A(u) \cdot \mu_B(u), u \in U$."

Definition 1.1.2 *[38] "Consider the fuzzy sets A, B, C and D over the universe of discourse U. The triangular norm (t-norm) is real-valued function from $[0,1] \times [0,1]$ to $[0,1]$ which satisfy the following conditions:*

(i) $t(0,0) = 0$, $t(\mu_A(u), 1) = t(1, \mu_A(u)) = \mu_A(u), u \in U$.

(ii) $t(\mu_A(u), \mu_B(u)) \leq t(\mu_C(u), \mu_D(u))$ *if* $\mu_A(u) \leq \mu_C(u)$ *and* $\mu_B(u) \leq \mu_D(u), u \in U$.

(iii) $t(\mu_A(u), \mu_B(u)) = t(\mu_B(u), \mu_A(u)), u \in U$.

(iv) $t(\mu_A(u), t(\mu_B(u), \mu_C(u))) = t(t(\mu_A(u), \mu_B(u)), \mu_C(u)), u \in U$.*"*

Definition 1.1.3 *[38] "Consider the fuzzy sets A, B, C and D over the universe of discourse U. The triangular conorm (t-conorm (s-norm)) is real-valued function from* $[0,1] \times [0,1]$ *to* $[0,1]$ *which satisfy the following conditions:*

(i) $s(1,1) = 1$, $s(\mu_A(u), 0) = s(0, \mu_A(u)) = \mu_A(u), u \in U$.

(ii) $s(\mu_A(u), \mu_B(u)) \leq s(\mu_C(u), \mu_B(u))$ *if* $\mu_A(u) \leq \mu_C(u)$ *and* $\mu_B(u) \leq \mu_D(u), u \in U$.

(iii) $s(\mu_A(u), \mu_B(u)) = s(\mu_B(u), \mu_A(u)), u \in U$.

(iv) $s(\mu_A(u), s(\mu_B(u), \mu_C(u))) = s(s(\mu_A(u), \mu_B(u)), \mu_C(u)), u \in U$.*"*

Fuzzy Relation and Composition Operators

Fuzzy relation is a mapping that maps the element through the cartesian product of one universe of discourse U with the another universe of discourse V to the unit interval $[0, 1]$. The strength of the relation in fuzzy environment between the ordered pair of the two universes is measured with the membership function expressing the different degrees of strength of the relation on the unit interval $[0, 1]$.

Definition 1.1.4 *[69] "A* ***fuzzy relation*** *R on fuzzy set U and V is a fuzzy subset of* $U \times V$*, i.e.,*

$$R = \{(u_1, u_2), \mu_R(u_1, u_2) \mid u_1 \in U, u_2 \in V\},$$

such that $\mu_R(u_1, u_2) \in [0, 1]$*. We denote* $FR(U \times V)$ *as a collection of all the fuzzy relations on* $U \times V$*."*

Definition 1.1.5 *[69] "Let R_1 and R_2 be the fuzzy relation on $U \times V$. Then the various binary operations are defined as follows:*

- ***Intersection:*** $\mu_{R\cap S}(u,v) = \min\{\mu_R(u,v), \mu_S(u,v)\}, (u,v) \in U \times V$.
- ***Union:*** $\mu_{R\cup S}(u,v) = \max\{\mu_R(u,v), \mu_S(u,v)\}, (u,v) \in U \times V$.
- ***Complement:*** $\mu_{\bar{R}}(u,v) = 1 - \mu_R(u,v), (u,v) \in U \times V$.
- ***Containment:*** $R \subset S \Rightarrow \mu_R(u,v) \leq \mu_S(u,v), (u,v) \in U \times V$."

Definition 1.1.6 *[69] "Suppose $R_1 \in FR(U \times V)$ and $R_2 \in FR(V \times Z)$ be two fuzzy relations. Then the various* ***composition operators*** *for the fuzzy relations R_1 and R_2 are defined as follows:*

- ***Max-Min Composition of Fuzzy Relations:*** *The $max - min$ composition relation of R_1 and R_2, denoted by $R_1 \circ R_2 \in FR(U \times Z)$, defined as*

$$R_1 \circ R_2 = \{(u,z), \mu_{R_1 \circ R_2}(u,z) \mid u \in U, z \in Z\},$$

where $\mu_{R_1 \circ R_2} = \max\{\min(\mu_{R_1}(u,v), \mu_{R_2}(v,z))\}\ v \in V$.

- ***Min-Max Composition of Fuzzy Relations:*** *The $min - max$ composition relation of R_1 and R_2, denoted by $R_1 \bullet R_2 \in FR(U \times Z)$, defined as*

$$R_1 \bullet R_2 = \{(u,z), \mu_{R_1 \bullet R_2}(u,z) \mid u \in U, z \in Z\},$$

where $\mu_{R_1 \bullet R_2} = \min\{\max(\mu_{R_1}(u,v), \mu_{R_2}(v,z))\}\ v \in V$.

- ***Max-Average Composition of Fuzzy Relations:*** *The $max-$ average composition relation of R_1 and R_2, denoted by $R_1 \Phi R_2 \in FR(U \times Z)$, defined as*

$$R_1 \Phi R_2 = \{(u,z), \mu_{R_1 \Phi R_2}(u,z) \mid u \in U, z \in Z\},$$

where $\mu_{R_1 \Phi R_2} = \frac{1}{2} \max\{\mu_{R_1}(u,v) + \mu_{R_2}(v,z)\}\ \ v \in V$.

- ***Min-Average Composition of Fuzzy Relations:*** *The $min-$ average composition relation of R_1 and R_2, denoted by $R_1 \Psi R_2 \in FR(U \times Z)$, defined as*

$$R_1 \Psi R_2 = \{(u,z), \mu_{R_1 \Psi R_2}(u,z) \mid u \in U, z \in Z\},$$

where $\mu_{R_1 \Psi R_2} = \frac{1}{2} \min\{\mu_{R_1}(u,v) + \mu_{R_2}(v,z)\}\ \ v \in V$."

1.1.2 Intuitionistic Fuzzy Set

Atanassov introduced the concept of intuitionistic fuzzy set (IFS)[67], which is an extension of the Zadeh's fuzzy set [68]. The IFS is characterized by membership function and non-membership function, which assign a value from the interval $[0,1]$ to every element in the sense of belongingness and non-belongingness respectively.

Definition 1.1.7 *[67] "Let U be the universe of discourse with $\mu_A : U \to [0,1]$ and $\nu_A : U \to [0,1]$ being the degree of membership and degree of non-membership respectively. The set $A = \{(u, \mu_u, \nu_u)|\ u \in U\}$ is called* **intuitionistic fuzzy set** *if it satisfies the condition $0 \leq \mu_A(u) + \nu_A(u) \leq 1$ with the degree of indeterminacy given by $\pi_A(u) = 1 - \mu_A(u) - \nu_A(u)$."*

Definition 1.1.8 *[67] "If $A, B \in IFS(U)$, then the standard binary operations can be defined as:*

*(a) **Complement:** $\overline{A} = \{< u, \nu_A(u), \mu_A(u) > \mid u \in U\}$;*

*(b) **Containment:** $A \subset B$ iff $\forall u \in U$, $\mu_A(u) \leq \mu_B(u)$ and $\nu_A(u) \geq \nu_B(u)$;*

*(c) **Union:** $A \cup B = \{< u, \mu_A(u) \vee \mu_B(u), \nu_A(u) \wedge \nu_B(u) > \mid u \in U\}$;*

*(d) **Intersection:** $A \cap B = \{< u, \mu_A(u) \wedge \mu_B(u), \nu_A(u) \vee \nu_B(u) > \mid u \in U\}$."*

1.1.3 Pythagorean Fuzzy Set

Yager [106] stated that the existing structures of FS and IFS are not capable enough to depict the human opinion in a broader sense and presented the following definition:

Definition 1.1.9 *[106] "A Pythagorean Fuzzy Set (PFS) M in U (universe of discourse) is given by*

$$M = \{< u, \mu_M(u), \nu_M(u) >|\ u \in U\};$$

where $\mu_M : U \to [0,1]$ and $\nu_M : U \to [0,1]$ represent the degree of membership and degree of non-membership respectively and for each $u \in U$ satisfy the condition

$$0 \leq \mu_M^2(u) + \nu_M^2(u) \leq 1.$$

The degree of indeterminacy for any Pythagorean fuzzy set M is given by $\pi_M(u) = \sqrt{1-\mu_M^2(u)-\nu_M^2(u)}\ \forall\ u \in U$."

The basic difference between PFS and IFS is the restriction corresponding to $\mu_M(u)$ and $\nu_M(u)$, i.e.,

$$0 \leq \mu_M^2(u) + \nu_M^2(u) \leq 1,$$

and

$$0 \leq \mu_M(u) + \nu_M(u) \leq 1$$

for $\mu_M(u), \nu_M(u) \in [0,1]$ respectively. The change in the constraint conditions is geometrically shown in the Figure 1.2. In this way, PFS can handle the uncertainty, impreciseness and vagueness in the information more efficiently and proves to be proficiently capable than IFS.

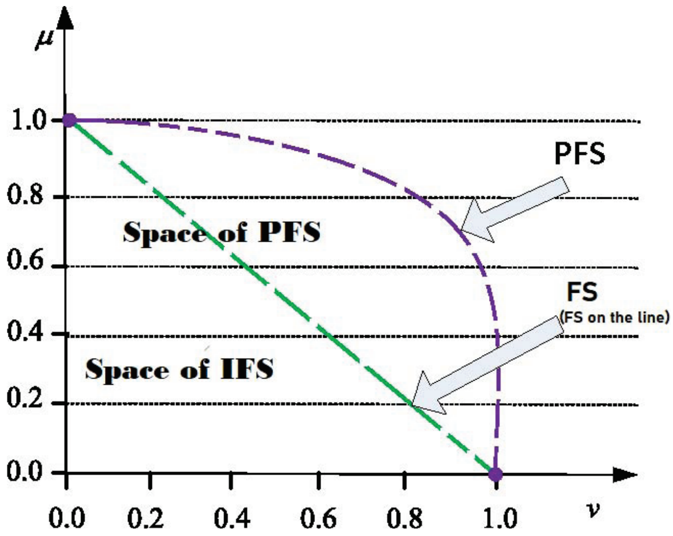

Figure 1.2: IFS vs PFS

Definition 1.1.10 *[141]* ***Binary Operations on PFSs***
"Consider $M = \{< u, \mu_M(u), \nu_M(u) >\mid u \in U\}$ and $N = \{< u, \mu_N(u), \nu_N(u) >\mid u \in U\}$ be two Pythagorean fuzzy sets over U (universe of discourse), then the operations can be defined as follows":

(a) $M^c = [\nu_M(u), \mu_M(u)],\ u \in U$.

(b) $M \cup N = \{\max(\mu_M(u), \mu_N(u)), \min(\nu_M(u), \nu_N(u))\},\ u \in U$.

(c) $M \cap N = \{\min(\mu_M(u), \mu_N(u)), \max(\nu_M(u), \nu_N(u))\},\ u \in U.$

(d) $M \cdot N = \{\mu_M(u) \cdot \mu_N(u), \nu_M(u) + \nu_N(u) - \nu_M(u) \cdot \nu_N(u)\},\ u \in U.$

(e) $M + N = \{\mu_M(u) + \mu_N(u) - \mu_M(u) \cdot \mu_N(u), \nu_M(u) \cdot \nu_N(u)\},\ u \in U.$

(f) $M \otimes N = \left\{\mu_M(u) \cdot \mu_N(u), \sqrt{(\nu_M(u))^2 + (\nu_N(u))^2 - (\nu_M(u))^2 \cdot (\nu_N(u))^2}\right\},\ u \in U.$

(g) $M \oplus N = \left\{\sqrt{(\mu_M(u))^2 + (\mu_N(u))^2 - (\mu_M(u))^2 \cdot (\mu_N(u))^2}, \nu_M(u) \cdot \nu_N(u)\right\},\ u \in U.$

Pythagorean fuzzy sets have been utilized by various researchers in order to deal with the various real world application fields such as decision-making problems, medical diagnosis, pattern recognition, etc. Based on score function, Zhang and Xu [139] presented a method to find the Pythagorean Fuzzy positive ideal solution (PIS) & the negative ideal solution (NIS) and also presented the extended version of TOPSIS method to determine the difference between each alternative with respect to PIS and NIS. A fused method between MOORA & PFSs for the selection of best/optimal alternative was stuided by Dominguez et al. [72]. Peng and Yang [142] presented some new kind of binary operations over PFSs and also studied various aggregation operators with their important properties. In continuation to this, they also provided an algorithm to solve group decision-making problems by using these proposed aggregation operators. Different kind of information measures for PFSs such as distance measure, similarity measure, entropy with inclusion measure, and their relations were studied by Peng et al. [141]. Further, to solve Pythagorean fuzzy MCDM problems, Zeng et al. [121] provided a new methodology by incorporating PFOWAWAD aggregation operator along with a hybrid TOPSIS method. Garg [52] proposed a correlation measure along with its weighted form in order to study the interaction between two PFSs. Wei and Wei [49] presented a similarity measure for PFS based on the cosine function to solve the problem of medical diagnosis and pattern recognition. Mohd and Abdullah [133] presented a new information measure for PFSs by using cosine similarity measures and Euclidean distance. Peng and Selvachandran [136] studied and presented the complete state-of-art related to the studies carried out in the field of PFSs and its applications with future directions. Xiao and Ding [44] provided divergence measure for PFSs for solving the medical diagnosis problem.

1.2 Literature Review

In this section, we have briefly reviewed the important, popular and widely used MCDM techniques under variable in the circumstances.

1.2.1 Multi-criteria Decision-Making Techniques

The objective of Multi-Criteria Decision-Making (MCDM) process is to achieve the best/optimal alternative from the available set of alternatives under the certain predefined set of criteria. In literature, various researchers have worked on the techniques for solving the MCDM problems. For example, Hwang and Yoon [21] proposed the "*Technique for Order Preference by Similarity to Ideal Solutions (TOPSIS)*" approach, Opricovic [118] developed the "*Vlsekriterijumska Optimizacija i Kompromisno Resenje (VIKOR)*" method, Brans and Mareschel [63] introduced the PROMETHEE ("*Preference Ranking Organization Method for Enrichment Evaluations*") method, Benayoun et al. [99] studied ELECTRE ("*elimination et choice translating reality*") method, Gomes and Lima [71] presented the TODIM ("*TOmada deDecisao Interativa e Multicriterio*") method and etc. Further, Opricovic and Tzeng [119] presented an extended version of VIKOR method by stating the limitations of TOPSIS, PROMOTHEE and ELECTRE methods. Because of the important feature of compromise solution, VIKOR method is more popular in research world than any other available method/technique.

The most decisive role in the MCDM problem is of the assignment of the criteria weights. The selection of the optimal solution/alternative depends on the proper assignment of the weights. Chen and Li [122] categorized the estimation of the criteria weights into two categories:

- First one is subjective evaluation, where the weights are concerned with the preference expressed by decision makers. Some examples of subjective weight category are as - Weighted least square method [12], Analytical Hierarchy Process (AHP) [125], Delphi method [19] and many more.

- The other category is objective evaluation where we determine weight by utilizing various techniques based on mathematical models such as - multi-objective programming method [41], principle element analysis [148], entropy method and etc.

Both of these two categories have their own merits and demerits. However, the entropy method of objective evaluation is highly trusted and utilized method to determine the criteria weights. The subjective evaluation is highly beneficial where there is no information loss and all the weights are available. But in many real world problems, there may be the cases where the information is not reliable due to some constraints such as time pressure, incomplete information about alternatives/criteria, limited expertise of the problem domain and etc. In such circumstances, the objective weights evaluation methods become more helpful and reliable.

1.2.2 Soft Set Theory to Soft Matrices

Many theories are found in the literature which have their own limitations to deal with the vagueness, uncertainty and impreciseness because of the involvement of parameterization tools presented in the different application fields of engineering, social/economic problems, decision-making problems etc. In order to overcome the above stated limitations, a new kind of mathematical tool has been developed by Molodtsov [27] (notion of soft set) to handel the vagueness, uncertainty and impreciseness in a better way. Next, in extension with the notion of soft set theory, Maji et al. [92][93][94] presented the "*fuzzy soft set (FSS)*" & "*intuitionistic fuzzy soft set (IFSS)*" along with their various standard binary operations and utilized them to solve the decision-making problems. The notion of Pythagorean fuzzy soft set (PFSS) along with various standard binary operators has been extended by the Peng et al. [140].

Further, Naim and Serdar [24] introduced the concept of soft matrices which are representations of the Molodtsov's soft sets and successfully applied the soft matrices in decision-making problems. Yong et al. [146] and Chetia et al. [13] extended the matrix representation of soft set to fuzzy soft set and intuitionistic fuzzy soft matrix respectively and applied it to decision-making problems.

Further, the mathematical generalization can be referred from [27], [24], [140] in the following way:

"*Let* $U = \{u_1, u_2, ..., u_m\}$ *be the universe of discourse and* $E = \{e_1, e_2, ..., e_n\}$ *be the set of parameters.*

- *The pair* (F, E) *is called soft set over* U *if and only if* $F : E \to \mathcal{P}(U)$*, where* $\mathcal{P}(U)$ *is the power set of* U*.*

- *Let* $F(U)$ *denotes the set of all fuzzy sets of* U*. A pair* (F, E) *is called a fuzzy soft set over* $F(U)$*, where* F_E *is a mapping given by* $F_E : E \to \mathcal{P}(F(U))$*.*

- *The pair* (F, E) *is called the Pythagorean Fuzzy Soft Set (PFSS) over* U *if* $F_E : E \to PFS(U)$ *and can be represented as*

$$(F, E) = \{(e, F(e)) : e \in E, F(e) \in PFS(U)\},$$

  where $PFS(U)$ *denotes the set of all Pythagorean fuzzy sets of* U*.*

- *Let* (F, E) *be a soft set over* U*. Then the subset* $U \times E$ *is uniquely defined by relation* $R_E = \{(u, e), e \in E, u \in U\}$*.*

  The characteristic function of R_E *is* $\chi_{R_E} : U \times E \to [0, 1]$ *given by*

$$\chi_{R_E}(u, e) = \begin{cases} 1 & if \;\; (u, e) \in U \times E \\ 0 & if \;\; (u, e) \notin U \times E \end{cases}.$$

  If $a_{ij} = \chi_{R_E}(u_i, e_j)$, then a matrix $[a_{ij}] = [\chi_{R_E}(u_i, e_j)]$ is called *soft matrix* of the soft set (F, E) over U of order $m \times n$."

1.2.3 Dimensionality Reduction Techniques

In order to convert a higher dimensional vector to a lower dimensional vector, the dimensionality reduction technique is utilized. The main objective of the dimensionality reduction techniques are to enhance the ability to handle irrelevant and redundant features, to enhance the cost efficiency and many more etc. In view of the decision processes, it will be difficult to visualize and work with a higher number of involved factors. Thus, the dimensionality reduction approach turns out to be an important study in different fields of application which have the extreme data modality. In the field of statistical science, many researchers have worked in the direction of dimensionality reduction by using various techniques such as "*Principal Component Analysis (PCA)*", "*Linear Discriminant Analysis (LDA)*" [1], "*singular value decomposition*" & "*learning vector quantization approach*" [79]. In the soft set theory, the concept of parameterization reduction has been presented by Chen et al. [29]. Xu et al. [143] provided the sequential and simultaneous perspectives approach for the reduction of data. Also, two new algorithms for the dimensionality reduction approach by using the concept of "*linear sequence discriminant analysis (LSDA)*" has been presented by Su et al. [15]. Further, incorporating the fuzzy transform method, a technique for the reduction of data has been proposed by the Perfilieva [57]. Konat et al. [1] presented a new technique for the reduction of the dimensionality of the original log set of Chinese Continental Scientific Drilling Main Hole to a convenient size by using the PCA and LDA. Sabitha et al. [79] used the three different kinds of dimensional reduction techniques, i.e., PCA, "*Singular Value decomposition*" & "*Learning Vector Quantization*" and applied these techniques to data set related to solar irradiance which comprises of temperature, solar irradiance, and humidity data. They also evaluate the efficiency and attain the best technique to be applicable for the data set. Chaterjee et al. [90] presented a hybrid method for the selection and evaluation of machining processes and utilized the pairwise comparison approach to estimate the weights in multi-criteria decision-making problem. In order to examine the consistency of results in the process of decision-making and to choose the optimal solution Mukhametzyanov and Pamucar [56] presented a mathematical MCDM model. In addition to this, they also carried out the sensitivity of the proposed model by using the different available methods, e.g., "*SAW, MOORA, VIKOR, COPRAS, CODAS, TOPSIS, D'IDEAL, MABAC, PROMETHEE-I,II, ORESTE-II*". By using the notion of fuzzy soft set Hooda and Kumari [35] proposed a new dimensionality reduction approach to solve decision making problem.

1.2.4 Entropy Measures

The concept of Entropy measure firstly coined by Shannon [16] in his famous paper "*The mathematical theory of communication*". The entropy measure is also known as the measure of information. It was introduced on the set of some finite number of probability distribution and also provided a mathematical model for establishing the concept of information measure. After Shannon's work various researchers have paid their interest in the development of information measures. This development was initiated by Renyi [10] with the inclusion of one parameter α. Havrda and Charvat [61] presented a non-additive entropy measure which was further generalized by Sharma and Mittal [14] by including two parameters and this new measure is known as entropy measure of order-α, type-β.

The first non-probabilistic entropy measure under fuzzy setup was studied by De Luca and Termini [3] which satisfies the four basic axioms: "*sharpness, maximality, symmetry and resolution*". Various researchers have introduced the different kinds of fuzzy entropy measures in order to solve various real life problems [28] [84] [60].

Definition 1.2.1 *[16] Let $\triangle_n = \{P = (p_1, p_2, \ldots, p_n),\ p_i \geq 0,\ i = 1, 2, 3, \ldots, n$ and $\sum p_i = 1\}$ be the set of all probability distribution association with random variable X taking finite values $x_1, x_{2,}, \ldots, x_n$. For any probability distribution $P = (p_1, p_2, \ldots, p_n) \in \triangle_n$, Shannon defined an entropy as:*

$$H(P) = -\sum_{i=1}^{n} (p_i) \log(p_i).$$

Definition 1.2.2 *[3] "The measure of fuzzy entropy between the fuzzy sets A and B is defined as a set-to-point mapping $H : FS(U) \longrightarrow \mathbb{R}^+$ which satisfies the following conditions:*

(i) $H(A) = 0$, if A is a crisp set in U;

(ii) $H(A)$ has a unique maximum value 1 if $\mu_A = \frac{1}{2}$;

(iii) $H(A) = H(A^c)$ if A^c is the complement of A;

(iv) $H(A) \leq H(B)$ *if* A *is less fuzzy then* B *i.e.,* $\mu_A \leq \mu_B$ *when* $\mu_B \leq \frac{1}{2}$ *and* $\mu_A \geq \mu_B$ *when* $\mu_B \geq \frac{1}{2}$*."*

Let A be the fuzzy set over the universe of discourse $U = \{u_1, u_2, \ldots, u_n\}$. The fuzzy entropy has been studied by many researchers and some of them are presented as follows:

- **Kaufman Fuzzy Entropy [8]**

$$H_K(A) = -\frac{1}{\ln n}\sum_{i=1}^{n} \Phi_A(u_i)\ln\Phi_A(u_i); \text{ where } \Phi_A(u_i) = \frac{\mu_A(u_i)}{\sum\limits_{i=1}^{n}\mu_A(u_i)}.$$

- **De Luca and Termini Fuzzy Entropy [3]**

$$H_D(A) = -\frac{1}{n\ln 2}\sum_{i=1}^{n}\big[\mu_A(u_i)\ln\mu_A(u_i) + (1-\mu_A(u_i))\ln(1-\mu_A(u_i))\big].$$

- **Renyi's Fuzzy Entropy [10]**

$$H_R(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\big[\mu_A^{\alpha}(u_i) + (1-\mu_A^{\alpha}(u_i))\big];\ \alpha \neq 1,\ \alpha > 0.$$

- **Pal and Pal Fuzzy Entropy [84]**

$$H_P(A) = \frac{1}{n\sqrt{e}-1}\sum_{i=1}^{n}\big[\mu_A(u_i)e^{1-\mu_A(u_i)} + (1-\mu_A(u_i))e^{\mu_A(u_i)} - 1\big].$$

After the effective applications of the IFS in various application fields, many researchers have studied and presented the entropy measures analogous to fuzzy entropy measures. Szmidt and Kacprzyk [39] extended the set of basic axioms of entropy measure from fuzzy set to intuitionistic fuzzy set. Based on De Luca and Termini fuzzy entropy [3], Zhang and Jiang [98] studied the entropy measure in intuitionistic fuzzy setup. Ye [65] proposed two entropy measures for IFSs. Verma and Sharma [110] presented an entropy measure based on exponential function under IFS environment. Many researchers have worked on the development of IFSs entropy measures [25] [86].

Definition 1.2.3 *"The measure of intuitionistic fuzzy entropy between the intuitionistic fuzzy sets* A *and* B *is defined mapping* $H : IFS(U) \longrightarrow \mathbb{R}^+$ *which satisfies the following conditions:*

(i) $H(A) = 0$, *if* A *is a crisp set in* U;

(ii) $H(A) = 1$, *if* $\mu_A = \nu_A$;

(iii) $H(A) = H(A^c)$ *if* A^c *is the complement of* A;

(iv) $H(A) \leq H(B)$ *if* $\mu_A \leq \mu_B$ *and* $\nu_A \geq \nu_A$."

Suppose A is an IFS over the universe of discourse $U = \{u_1, u_2, \ldots, u_n\}$. The intuitionistic entropy measure has been studied by many researchers in the literature and some of them are presented as follows:

- **Vlachos and Sergiadis IF Entropy Measure [53]**

$$H(A) = -\frac{1}{\ln 2}\sum_{i=1}^{n}\Bigg[\mu_A(u_i)\log\mu_A(u_i) + \nu_A(u_i)\log\nu_A(u_i) - \Big((1-\pi_A(u_i))\log(1-\pi_A(u_i))\Big) - \pi_A(u_i)\log 2\Bigg];$$

- **Zhang and Jiang IF Entropy Measure [98]**

$$H(A) = -\frac{1}{n}\sum_{i=1}^{n}\Bigg[\left(\frac{\mu_A(u_i)+1-\nu_A(u_i)}{2}\right)\log\left(\frac{\mu_A(u_i)+1-\nu_A(u_i)}{2}\right) + \Big(\frac{\nu_A(u_i)+1-\mu_A(u_i)}{2}\Big)\log\left(\frac{\nu_A(u_i)+1-\mu_A(u_i)}{2}\right)\Bigg].$$

- **Wei et al. IF Entropy Measure [25]**

$$H(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\sqrt{2}}{\sqrt{2}-1}\left(\cos\pi\left(\frac{\mu_A(u_i)-\nu_A(u_i)}{4}\right)-1\right)\right].$$

1.2.5 Discriminant Measures

The divergence/discriminant measure is an information measure which is also known as the relative entropy measure and gives a difference formula between the two discrete probability distributions. Bhandari and Pal [28] studied and extended the Kullback and Leibler's [114] divergence measure over fuzzy environment based on the mutual information measure. Based on exponential function, Fan and Xie [60] proposed a divergence measure and studied its relation with the fuzzy exponential entropy. Next, Montes et al. [116] discussed the special classes of divergence measures in connection with fuzzy and probabilistic uncertainty. Further, the fuzzy divergence measure have been successfully implemented by Ghosh et al. [75] to study the automated leukocyte recognition. Bhatia and Singh [95] proposed four different type of fuzzy directed divergence measures.

Analogous to Shang & Jiang [137] discriminant measure, Vlachos and Sergiadis [53] provided the discriminant measure for intuitionistic fuzzy setup. Further, Wang et al. [135] and Hung et al. [132] presented a set of axioms for the distance measure and for the discriminant measure respectively. Li [31] provided the intuitionistic fuzzy discriminant measure and Hung et al. [131] proposed J-divergence measure between intuitionistic fuzzy sets with their application in pattern recognition. Montes et al. [55] established some important relationships among divergence measures, dissimilarity measures and distance measures. Analogous to the basic fuzzy discriminant measures, intuitionistic fuzzy discriminant measures exhibits wider applications in various application fields such as decision-making problems ([100], [32], [108], [109], [113]), medical diagnosis ([115], [98], [6]), logical reasoning [144], linguistic variables [132] and pattern recognition ([4], [131], [134], [46], [53]) etc.

Kaya and Kahraman [54] have provided comparison of fuzzy multi-criteria decision making methods for intelligent building assessment along with detailed ranking results. Bajaj et al. [104] proposed a new R-norm intuitionistic fuzzy entropy and a weighted R-norm Intuitionistic fuzzy divergence measure with their computational applications in pattern recognition and image thresholding. Gandotra et al. [83] studied multiple-criteria decision making problem with the help of parametric entropy under α-cut and (α, β)-cut based distance measures for different possible values of parameters and provided the ranking method for the available alternatives.

Let A be the intuitionistic fuzzy set over the universe of discourse $U = \{u_1, u_2, \ldots, u_n\}$. The intuitionistic discriminant/divergence measure has been studied by many researchers in the literature and some of them are presented as follows:

- **Vlachos and Sergiadis IF Divergence Measure [53]**

$$I(A,B) = \frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(u_i)\log\frac{2\mu_A(u_i)}{\mu_A(u_i)+\mu_B(u_i)} + \nu_A(u_i)\log\frac{2\nu_A(u_i)}{\nu_A(u_i)+\nu_B(u_i)}\right];$$

- **Verma and Sharma IF Divergence Measure [108]**

$$H(A,B) = \frac{1}{n}\sum_{i=1}^{n}\Bigg[\mu_A(u_i)\log\left(\frac{\mu_A(u_i)}{\lambda\mu_A(u_i)+(1-\lambda)\mu_B(u_i)}\right) + \nu_A(u_i)\log\left(\frac{\nu_A(u_i)}{\lambda\nu_A(u_i)+(1-\lambda)\nu_B(u_i)}\right) + \pi_A(u_i)\log\left(\frac{\pi_A(u_i)}{\lambda\pi_A(u_i)+(1-\lambda)\pi_B(u_i)}\right)\Bigg].$$

1.3 Motivation

The way we think, process information, make our decision by particularly involving our perception, language, human opinion, fuzziness is very inherited and such situations can be found everywhere. The best way to deal with such situations is to deploy the theory of fuzzy set which is characterized by a membership function. For the sake of covering the imprecise information in a better way, Atanassov extended this notion of fuzzy set to intuitionistic fuzzy set which was characterized by its membership function & non-membership function. Further, R. R. Yager extended the restriction on the constraint by introducing a new set called as Pythagorean fuzzy set. While doing literature survey, we found that:

- Pythagorean fuzzy set seems to be the more generalized fuzzy set and have the wider coverage of information span so that the decision-making process can be dealt more effectively.
- No study has been carried out by utilizing the Pythagorean fuzzy setup together with the notion of soft matrices and applications.
- No dimensionality reduction technique is available in the literature to reduce the informational data in the Pythagorean setup.
- No study was presented regarding the Pythagorean fuzzy entropy and discriminant information measures in the available literature.

Chapter 2

Pythagorean Fuzzy Soft Matrices

In this chapter, the notion of Pythagorean fuzzy soft matrix (PFSM) and various applications in the field of decision-making and medical diagnosis have been presented and studied in detail. Different types of PFSMs and several related binary operations have been presented with important properties. By analogously incorporating the concept of choice matrix and weighted choice matrix, an algorithm for solving decision-making problem has been provided along with an illustrating example. Further, an algorithm to deal with a general medical diagnosis problem has also been provided by using the definitions of score/utility matrix along with the demonstration of the numerical example. A detailed comparative analysis has also been carried out for better understanding.

2.1 PFSMs and its Binary Operations

The notion of matrices significantly helps in various soft computing applications and in handling the dimensionality feature of the big data problems related to various engineering problems. In view of the important role of matrices, we present the notion of PFSMs along with different binary operations.

Definition 2.1.1 *Let (F, E) be a Pythagorean fuzzy soft set over X, then the subset $X \times E$ is uniquely defined by $R_E = \{(x, e), e \in E, x \in X\}$. The R_E can be characterized by its membership function and non membership function given by $\mu_{R_E} : X \times E \to [0, 1]$*

and $\nu_{R_E} : X \times E \to [0,1]$ *respectively.*
If $(\mu_{ij}, \nu_{ij}) = (\mu_{R_E}(x_i, e_j), \nu_{R_E}(x_i, e_j))$*, where* $\mu_{R_E}(x_i, e_j)$ *is the membership of* x_i *in the Pythagorean fuzzy set* $F(e_j)$ *and* $\nu_{R_E}(x_i, e_j)$ *is the non-membership of* x_i *in the Pythagorean fuzzy set* $F(e_j)$ *respectively, then we define a matrix given by*

$$[M] = [m_{ij}]_{m\times n} = [(\mu_{ij}^M, \nu_{ij}^M)]_{m\times n} = \begin{bmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & \cdots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m1}, \nu_{m1}) & (\mu_{m2}, \nu_{m2}) & \cdots & (\mu_{mn}, \nu_{mn}) \end{bmatrix}$$

which is called ***Pythagorean fuzzy soft matrix*** *of order* $m \times n$ *over* X*.*

For a proper understanding of the construction of a PFSM, let us consider a universe of discourse $X = \{x_1, x_2, x_3\}$ with a parameter set $(E = \{e_1, e_2, e_3\})$ and

$$F(e_1) = \{(x_1, 0.6, 0.5), (x_2, 0.5, 0.8), (x_3, 0.9, 0.2)\},$$
$$F(e_2) = \{(x_1, 0.8, 0.5), (x_2, 0.9, 0.3), (x_3, 0.6, 0.6)\},$$
$$F(e_3) = \{(x_1, 0.6, 0.7), (x_2, 0.5, 0.6), (x_3, 0.7, 0.5)\}.$$

We take the soft set (F, E) given by $F(e_1), F(e_2), F(e_3)$ over the universe of discourse.

In this way, we can write the PFSM $[M(F, E)]$ as

$$[M] = [(\mu_{ij}^M, \nu_{ij}^M)]_{m\times n} = \begin{bmatrix} (0.6, 0.5) & (0.8, 0.5) & (0.6, 0.7) \\ (0.5, 0.8) & (0.9, 0.3) & (0.5, 0.6) \\ (0.9, 0.2) & (0.6, 0.6) & (0.7, 0.5) \end{bmatrix}.$$

Definition 2.1.2 ***Various kinds of Pythagorean fuzzy soft matrices:*** *Suppose* $PFSM_{m\times n}$ *is a collection of all Pythagorean fuzzy soft matrices over* X*. A Pythagorean fuzzy soft matrix* $M = [(\mu_{ij}^M, \nu_{ij}^M)] \in PFSM_{m\times n}$ *is called:*

- ***Pythagorean fuzzy soft zero matrix*** *if*

$$\mu_{ij}^M = 0 \text{ and } \nu_{ij}^M = 0; \forall i, j \text{ and is denoted by } 0 = [0, 0].$$

- ***Pythagorean fuzzy soft square matrix*** *if* $m = n$*.*

- ***Pythagorean fuzzy soft row matrix*** *if* $n = 1$*.*

- ***Pythagorean fuzzy soft column matrix*** *if* $m = 1$.

- ***Pythagorean fuzzy soft diagonal matrix*** *if all its non-diagonal elements are zero* $\forall$ i, j.

- ***Pythagorean fuzzy soft μ-universal matrix*** *if* $\mu_{ij}^{M} = 1$ *and* $\nu_{ij}^{M} = 0$ $\forall$ i *and* j, *denoted by* P_{μ}.

- ***Pythagorean fuzzy soft ν-universal matrix*** *if* $\mu_{ij}^{M} = 0$ *and* $\nu_{ij}^{M} = 1$ $\forall$ i *and* j, *denoted by* P_{ν}.

- ***Scalar multiplication of Pythagorean fuzzy soft matrix :*** *for any scalar* k, *we define* $kA = [(k\mu_{ij}^{M}, k\nu_{ij}^{M})]$, $\forall$ i *and* j.

Definition 2.1.3 ***Relations over Pythagorean fuzzy soft matrices:***
Consider two Pythagorean fuzzy soft matrices $M = [(\mu_{ij}^{M}, \nu_{ij}^{M})]$ *and* $N = [(\mu_{ij}^{N}, \nu_{ij}^{N})] \in PFSM_{m\times n}$. *Then the relations over two Pythagorean fuzzy soft matrices is called:*

- ***Sub matrix:*** $M \subseteq N$ *if* $\mu_{ij}^{M} \leq \mu_{ij}^{N}$ *and* $\nu_{ij}^{M} \geq \nu_{ij}^{N}$ $\forall$ i *and* j.

- ***Super matrix:*** $M \supseteq N$ *if* $\mu_{ij}^{M} \geq \mu_{ij}^{N}$ *and* $\nu_{ij}^{M} \leq \nu_{ij}^{N}$ $\forall$ i *and* j.

- ***Equal matrix:*** $M = N$ *if* $\mu_{ij}^{M} = \mu_{ij}^{N}$ *and* $\nu_{ij}^{M} = \nu_{ij}^{N}$ $\forall$ i *and* j.

- ***Max Min Product of Pythagorean fuzzy soft matrix:***
  Let $M = [a_{ij}] = [(\mu_{ij}^{M}, \nu_{ij}^{M})] \in PFSM_{m\times n}$ *&* $N = [b_{jk}] = [(\mu_{jk}^{N}, \nu_{jk}^{N})] \in PFSM_{n\times p}$ *be two Pythagorean fuzzy soft matrices then*
  $M * N = [c_{ik}]_{m\times p} = \left[\left\{\max(\min_{j}(\mu_{ij}^{M}, \mu_{jk}^{N})), \min(\max_{j}(\nu_{ij}^{M}, \nu_{jk}^{N}))\right\}\right]$ $\forall$ i, j *and* k.

Definition 2.1.4 ***Operations over Pythagorean Fuzzy Soft Matrices:***
Consider two Pythagorean fuzzy soft matrices $A = [(\mu_{ij}^{A}, \nu_{ij}^{A})]$ *and* $B = [(\mu_{ij}^{B}, \nu_{ij}^{B})] \in PFSM_{m\times n}$. *Then various standard operations over two Pythagorean fuzzy soft matrices can be defined as follows:*

- $A^{c} = \left[(\nu_{ij}^{A}, \mu_{ij}^{A})\right]$ $\forall$ i *and* j.

- $A \cup B = \left[\max(\mu_{ij}^{A}, \mu_{ij}^{B}), \min(\nu_{ij}^{A}, \nu_{ij}^{B})\right]$ $\forall$ i *and* j.

- $A \cap B = \left[\min(\mu_{ij}^{A}, \mu_{ij}^{B}), \max(\nu_{ij}^{A}, \nu_{ij}^{B})\right]$ $\forall$ i *and* j.

- $A \cdot B = \left[(\mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A + \nu_{ij}^B - \nu_{ij}^A \cdot \nu_{ij}^B)\right]$ $\forall$ *i and j.*

- $A + B = \left[(\mu_{ij}^A + \mu_{ij}^B - \mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A \cdot \nu_{ij}^B)\right]$ $\forall$ *i and j.*

- $A \otimes B = \left[(\mu_{ij}^A \cdot \mu_{ij}^B, \sqrt{(\nu_{ij}^A)^2 + (\nu_{ij}^B)^2 - (\nu_{ij}^A)^2 \cdot (\nu_{ij}^B)^2})\right]$ $\forall$ *i and j.*

- $A \oplus B = \left[(\sqrt{(\mu_{ij}^A)^2 + (\mu_{ij}^B)^2 - (\mu_{ij}^A)^2 \cdot (\mu_{ij}^B)^2}, \nu_{ij}^A \cdot \nu_{ij}^B)\right]$ $\forall$ *i and j.*

- $A@B = \left[(\frac{\mu_{ij}^A + \mu_{ij}^B}{2}, \frac{\nu_{ij}^A + \nu_{ij}^B}{2})\right]$ $\forall$ *i and j.*

- $A@_w B = \left[(\frac{w_1\mu_{ij}^A + w_2\mu_{ij}^B}{w_1+w_2}, \frac{w_1\nu_{ij}^A + w_2\nu_{ij}^B}{w_1+w_2})\right]$ $\forall$ *i and j ; where* $w_1, w_2 > 0$ *are the weights.*

- $A\$B = \left[\left(\sqrt{\mu_{ij}^A \cdot \mu_{ij}^B}, \sqrt{\nu_{ij}^A \cdot \nu_{ij}^B}\right)\right]$ $\forall$ *i and j.*

- $A\$_w B = \left[((\mu_{ij}^A)^{w_1} \cdot (\mu_{ij}^B)^{w_2})^{\frac{1}{w_1+w_2}}, ((\nu_{ij}^A)^{w_1} \cdot (\nu_{ij}^B)^{w_2})^{\frac{1}{w_1+w_2}})\right]$ $\forall$ *i and j, where* $w_1, w_2 > 0$ *are the weights.*

- $A \bowtie B = \left[(2 \cdot \frac{\mu_{ij}^A \cdot \mu_{ij}^B}{\mu_{ij}^A + \mu_{ij}^B}, 2 \cdot \frac{\nu_{ij}^A \cdot \nu_{ij}^B}{\nu_{ij}^A + \nu_{ij}^B})\right]$ $\forall$ *i and j.*

- $A \bowtie_w B = \left[(\frac{w_1+w_2}{\frac{w_1}{\mu_{ij}^A} + \frac{w_2}{\mu_{ij}^B}}, \frac{w_1+w_2}{\frac{w_1}{\nu_{ij}^A} + \frac{w_2}{\nu_{ij}^B}})\right]$ $\forall$ *i and j ; where* $w_1, w_2 > 0$ *are the weights.*

Proposition 2.1 *Suppose A and B* $\in PFSM_{m\times n}$ *are two Pythagorean fuzzy soft matrices then the following results hold:*

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

(iii) $A + B = B + A$

(iv) $A \cdot B = B \cdot A$

(v) $(A \cup B)^c = A^c \cap B^c$

(vi) $(A \cap B)^c = A^c \cup B^c$

(vii) $(A^c \cap B^c)^c = A \cup B$

(viii) $(A^c \cup B^c)^c = A \cap B$

(ix) $(A^c + B^c)^c = A \cdot B$

(x) $(A^c \cdot B^c)^c = A + B$.

Proof : Let $A = [(\mu_{ij}^A, \nu_{ij}^A)], B = [(\mu_{ij}^B, \nu_{ij}^B)] \in PFSM_{m\times n}$.

For each values of i & j,

(i) $A \cup B = [\max(\mu_{ij}^A, \mu_{ij}^B), \min(\nu_{ij}^A, \nu_{ij}^B)] = [\max(\mu_{ij}^B, \mu_{ij}^A), \min(\nu_{ij}^B, \nu_{ij}^A)] = B \cup A.$

(ii) $A \cap B = [\min(\mu_{ij}^A, \mu_{ij}^B), \max(\nu_{ij}^A, \nu_{ij}^B)] = [\min(\mu_{ij}^B, \mu_{ij}^A), \max(\nu_{ij}^B, \nu_{ij}^A)] = B \cap A.$

(iii) $A + B = [(\mu_{ij}^A + \mu_{ij}^B - \mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A \cdot \nu_{ij}^B)] = [(\mu_{ij}^B + \mu_{ij}^A - \mu_{ij}^B \cdot \mu_{ij}^A, \nu_{ij}^B \cdot \nu_{ij}^A)] = B + A.$

(iv) $A \cdot B = [(\mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A + \nu_{ij}^B - \nu_{ij}^A \cdot \nu_{ij}^B)] = [(\mu_{ij}^B \cdot \mu_{ij}^A, \nu_{ij}^B + \nu_{ij}^A - \nu_{ij}^B \cdot \nu_{ij}^A)] = B \cdot A.$

(v) $(A \cup B)^c = ([(\mu_{ij}^A, \nu_{ij}^A)] \cup [(\mu_{ij}^B, \nu_{ij}^B)])^c = [\max(\mu_{ij}^A, \mu_{ij}^B), \min(\nu_{ij}^A, \nu_{ij}^B)]^c$

$= [\min(\nu_{ij}^A, \nu_{ij}^B), \max(\mu_{ij}^A, \mu_{ij}^B)] = [(\nu_{ij}^A, \mu_{ij}^A)] \cap [(\nu_{ij}^B, \mu_{ij}^B)] = A^c \cap B^c.$

On similar lines, the proof of $(vi) - (x)$ can be carried out.

Proposition 2.2 *If $A = [(\mu_{ij}^A, \nu_{ij}^A)] \in PFSM_{m\times n}$ then the following results can be verified in accordance with the definition:*

(i) $(A^c)^c = A$

(ii) $(P_\mu)^c = P_\nu$

(iii) $(P_\nu)^c = P_\mu$

(iv) $A \cup A = A$

(v) $A \cup P_\mu = P_\mu$

(vi) $A \cap P_\nu = A$

(vii) $A \cap A = A$

(viii) $A \cap P_\mu = A$

(ix) $A \cap P_\nu = P_\nu.$

Proposition 2.3 *Suppose A & $B \in PFSM_{m\times n}$. The results related to the weighed operations hold:*

(i) $(A^c @_w B^c)^c = A @_w B$

(ii) $(A^c \$_w B^c)^c = A \$_w B$

(iii) $(A^c \bowtie_w B^c)^c = A \bowtie_w B$

(iv) $A @_w B = B @_w A$

(v) $A \$_w B = B \$_w A$

(vi) $A \bowtie_w B = B \bowtie_w A.$

Proof : Let $A = [(\mu_{ij}^A, \nu_{ij}^A)], B = [(\mu_{ij}^B, \nu_{ij}^B)] \in PFSM_{m\times n}$.

For each value of i, j & $w_1, w_2 > 0$, we have,

(i) $(A^c @_w B^c)^c = \left(\left[(\nu_{ij}^A, \mu_{ij}^A) @_w (\nu_{ij}^B, \mu_{ij}^B)\right]\right)^c = \left(\left[\frac{w_1\nu_{ij}^A + w_2\nu_{ij}^B}{w_1+w_2}, \frac{w_1\mu_{ij}^A + w_2\mu_{ij}^B}{w_1+w_2}\right]\right)^c$
$= \left[\frac{w_1\mu_{ij}^A + w_2\mu_{ij}^B}{w_1+w_2}, \frac{w_1\nu_{ij}^A + w_2\nu_{ij}^B}{w_1+w_2}\right] = A @_w B.$

(ii)

$$(A^c \$_w B^c)^c = \left([(\nu_{ij}^A, \mu_{ij}^A) \$_w (\nu_{ij}^B, \mu_{ij}^B)]\right)^c = \left(\left[((\nu_{ij}^A)^{w_1} \cdot (\nu_{ij}^B)^{w_2})^{\frac{1}{w_1+w_2}}, ((\mu_{ij}^A)^{w_1} \cdot (\mu_{ij}^B)^{w_2})^{\frac{1}{w_1+w_2}}\right]\right)^c$$
$$= \left[((\mu_{ij}^A)^{w_1} \cdot (\mu_{ij}^B)^{w_2})^{\frac{1}{w_1+w_2}}, ((\nu_{ij}^A)^{w_1} \cdot (\nu_{ij}^B)^{w_2})^{\frac{1}{w_1+w_2}}\right] = A \$_w B.$$

Similar proof for (iii).

(iv)

$$A @_w B = \left[\frac{w_1\mu_{ij}^A + w_2\mu_{ij}^B}{w_1+w_2}, \frac{w_1\nu_{ij}^A + w_2\nu_{ij}^B}{w_1+w_2}\right]$$
$$= \left[\frac{w_2\mu_{ij}^B + w_1\mu_{ij}^A}{w_2+w_1}, \frac{w_2\nu_{ij}^B + w_1\nu_{ij}^A}{w_2+w_1}\right] = B @_w A.$$

(v)

$$A \$_w B = \left[((\mu_{ij}^A)^{w_1} \cdot (\mu_{ij}^B)^{w_2})^{\frac{1}{w_1+w_2}}, ((\nu_{ij}^A)^{w_1} \cdot (\nu_{ij}^B)^{w_2})^{\frac{1}{w_1+w_2}}\right]$$
$$= \left[((\mu_{ij}^B)^{w_2} \cdot (\mu_{ij}^A)^{w_1})^{\frac{1}{w_2+w_1}}, ((\nu_{ij}^B)^{w_2} \cdot (\nu_{ij}^A)^{w_1})^{\frac{1}{w_2+w_1}}\right] = B \$_w A$$

Similar proof for (vi).

Proposition 2.4 *Suppose A, B & C $\in PFSM_{m\times n}$. The important results in connection with associativity of operations are as follows:*

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(iii) $(A + B) + C = A + (B + C)$

(iv) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

(v) $(A @ B) @ C = A @ (B @ C)$

(vi) $(A\$B)\$C = A\$(B\$C)$

(vii) $(A \bowtie B) \bowtie C = A \bowtie (B \bowtie C)$.

Proof : For each and every i & j, we get

(i)

$$(A \cup B) \cup C = [(\max\{\mu_{ij}^A, \mu_{ij}^B\}, \min\{\nu_{ij}^A, \nu_{ij}^B\})] \cup [(\mu_{ij}^C, \nu_{ij}^C)]$$
$$= [(\max\{(\mu_{ij}^A, \mu_{ij}^B), \mu_{ij}^C\}, \min\{(\nu_{ij}^A, \nu_{ij}^B), \nu_{ij}^C\})]$$
$$= [(\max\{(\mu_{ij}^A, (\mu_{ij}^B, \mu_{ij}^C))\}, \min\{\nu_{ij}^A, (\nu_{ij}^B, \nu_{ij}^C)\})] = A \cup (B \cup C).$$

(ii)

$$(A \cap B) \cap C = [(\min\{\mu_{ij}^A, \mu_{ij}^B\}, \max\{\nu_{ij}^A, \nu_{ij}^B\}] \cup (\mu_{ij}^C, \nu_{ij}^C)]$$
$$= [(\min\{(\mu_{ij}^A, \mu_{ij}^B), \mu_{ij}^C\}, \max\{(\nu_{ij}^A, \nu_{ij}^B), \nu_{ij}^C\})]$$
$$= [(\min\{(\mu_{ij}^A, (\mu_{ij}^B, \mu_{ij}^C))\}, \max\{\nu_{ij}^A, (\nu_{ij}^B, \nu_{ij}^C)\})] = A \cap (B \cap C).$$

(iii)

$$(A + B) + C = [(\mu_{ij}^A + \mu_{ij}^B - \mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A \cdot \nu_{ij}^B)] + [(\mu_{ij}^C, \nu_{ij}^C)]$$
$$= [(\mu_{ij}^A + \mu_{ij}^B) + \mu_{ij}^C - (\mu_{ij}^A \cdot \mu_{ij}^B) \cdot \mu_{ij}^C, (\nu_{ij}^A \cdot \nu_{ij}^B) \cdot \nu_{ij}^C]$$
$$= [\mu_{ij}^A + (\mu_{ij}^B + \mu_{ij}^C) - \mu_{ij}^A \cdot (\mu_{ij}^B \cdot \mu_{ij}^C), \nu_{ij}^A \cdot (\nu_{ij}^B \cdot \nu_{ij}^C)] = A + (B + C).$$

On similar lines, the proof of $(iv) - (vii)$ can be carried out.

Proposition 2.5 *Let A, B and $C \in PFSM_{m\times n}$ be three Pythagorean fuzzy soft matrices then the following results related to distributivity of operations hold:*

Proof : For each and every i & j, we have

(i)

$$A \cap (B \cup C) = [(\mu_{ij}^A, \nu_{ij}^A)] \cap [(\max\{\mu_{ij}^B, \mu_{ij}^C\}, \min\{\nu_{ij}^B, \nu_{ij}^C\})]$$
$$= [(\min\{\mu_{ij}^A, \max\{\mu_{ij}^B, \mu_{ij}^C\}\}, \max\{\nu_{ij}^A, \min\{\nu_{ij}^B, \nu_{ij}^C\}\})].$$

(i) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

(ii) $(A\cap B)\cup C=(A\cup C)\cap(B\cup C)$

(iii) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

(iv) $(A\cup B)\cap C=(A\cap C)\cup(B\cap C)$

(v) $(A\cap B)@C=(A@C)\cap(B@C)$

(vi) $(A\cap B)\bowtie C=(A\bowtie C)\cap(B\bowtie C)$

(vii) $(A\cup B)+C=(A+C)\cup(B+C)$

(viii) $(A\cup B)\cdot C=(A\cdot C)\cup(B\cdot C)$

(ix) $A\cup(B@C)=(A\cup B)@(A\cup C)$

(x) $(A\cup B)\bowtie C=(A\bowtie C)\cup(B\bowtie C)$

(xi) $A@(B\cup C)=(A@B)\cup(A@C)$

(xii) $A@(B\cap C)=(A@B)\cap(B@C)$

(xiii) $A\$(B\cup C)=(A\$B)\cup(A\$C)$

(xiv) $(A\cup B)\$C=(A\$C)\cup(B\$C)$

(xv) $A\cdot(B@C)=(A\cdot B)@(A\cdot C)$

(xvi) $A\cup(B\bowtie C)=(A\cup B)\bowtie(A\cup C)$

(xvii) $A\bowtie(B\cup C)=(A\bowtie B)\cup(A\bowtie C)$

(xviii) $A\$(B\cap C)=(A\$B)\cap(B\$C)$

(xix) $(A\cap B)\$C=(A\$C)\cap(B\$C)$.

Now,

$$\begin{aligned}(A\ \cap B)\cup(A\cap C)&=[(\min\{\mu_{ij}^{A},\mu_{ij}^{B}\},\max\{\nu_{ij}^{A},\nu_{ij}^{B}\}]\cup[(\min\{\mu_{ij}^{A},\mu_{ij}^{C}\},\max\{\nu_{ij}^{A},\nu_{ij}^{C}\}]\\&=[\max(\min\{\mu_{ij}^{A},\mu_{ij}^{B}\},\min\{\mu_{ij}^{A},\mu_{ij}^{C}\}),\min(\max\{\nu_{ij}^{A},\nu_{ij}^{B}\},\max\{\nu_{ij}^{A},\nu_{ij}^{C}\})]\\&=[\max(\mu_{ij}^{A},\min\{\mu_{ij}^{B},\mu_{ij}^{C}\}),\min(\nu_{ij}^{A},\max\{\nu_{ij}^{B},\nu_{ij}^{C}\})]\\&=[\min(\mu_{ij}^{A},\max\{\mu_{ij}^{B},\mu_{ij}^{C}\}),\max(\nu_{ij}^{A},\min\{\nu_{ij}^{B},\nu_{ij}^{C}\})]=A\cap(B\cup C).\end{aligned}$$

Hence, $A\ \cap(B\cup C)\ =\ (A\cap B)\cup(A\cap C)$ holds.

(ii)

$$\begin{aligned}(A\cap B)\ \cup C&=[(\min\{\mu_{ij}^{A},\mu_{ij}^{B}\},\max\{\nu_{ij}^{A},\nu_{ij}^{B}\}]\cup[(\mu_{ij}^{C},\nu_{ij}^{C})]\\&=[\max(\min\{\mu_{ij}^{A},\mu_{ij}^{B}\},\mu_{ij}^{C}),\min(\max\{\nu_{ij}^{A},\nu_{ij}^{B}\},\nu_{ij}^{C}].\end{aligned}$$

Now,

$$\begin{aligned}(A\cup C)\ \cap(B\cup C)&=[\max\{\mu_{ij}^{A},\mu_{ij}^{C}\},\min\{\nu_{ij}^{A},\nu_{ij}^{C}\}]\cap[\max\{\mu_{ij}^{B},\mu_{ij}^{C}\},\min\{\nu_{ij}^{B},\nu_{ij}^{C}\}]\\&=[\min(\max\{\mu_{ij}^{A},\mu_{ij}^{C}\},\max\{\mu_{ij}^{B},\mu_{ij}^{C}\}),\max(\min\{\nu_{ij}^{A},\nu_{ij}^{C}\},\min\{\nu_{ij}^{B},\nu_{ij}^{C}\})]\\&=[\min(\max\{\mu_{ij}^{A},\mu_{ij}^{B}\},\mu_{ij}^{C}\}),\max(\min\{\nu_{ij}^{A},\nu_{ij}^{B}\},\nu_{ij}^{C}\})]\\&=[\max(\min\{\mu_{ij}^{A},\mu_{ij}^{B}\},\mu_{ij}^{C}\}),\min(\max\{\nu_{ij}^{A},\nu_{ij}^{B}\},\nu_{ij}^{C}\})]=(A\cap\ B)\cup\ C\end{aligned}$$

Hence, $(A\ \cap B)\cup C\ =\ (A\ \cup C)\cap(B\ \cup C)$.

Similarly, the results $(iii)-(xix)$ can be established.

2.2 Applications in Decision Making

Here, we present an algorithm (Figure 2.1) to solve the decision-making problem by taking the idea of PFSM into account. For this, we first proposed the revised definition of choice matrix and its weighted form as follows:

Definition 2.2.1 *Consider $A = [(\mu_{ij}^A, \nu_{ij}^A)] \in PFSM_{m\times n}$ be a Pythagorean fuzzy soft matrix, then the choice matrix of matrix A is*

$$C(\ A\ ) = \left[\left(\frac{\sum_{j=1}^{n}(\mu_{ij}^A)^2}{n}, \frac{\sum_{j=1}^{n}(\nu_{ij}^A)^2}{n}\right)\right]_{m\times 1} \quad \forall\ i \text{ when weights are equal.}$$

Definition 2.2.2 *Consider $A = [(\mu_{ij}^A, \nu_{ij}^A)] \in PFSM_{m\times n}$ be a Pythagorean fuzzy soft matrix, then the weighted choice matrix of matrix A is given by*

$$C_w(\ A\ ) = \left[\left(\frac{\sum_{j=1}^{n} w_j(\mu_{ij}^A)^2}{\sum w_j}, \frac{\sum_{j=1}^{n} w_j(\nu_{ij}^A)^2}{\sum w_j}\right)\right]_{m\times 1} \quad \forall\ i \text{ where } w_j > 0 \text{ are weights.}$$

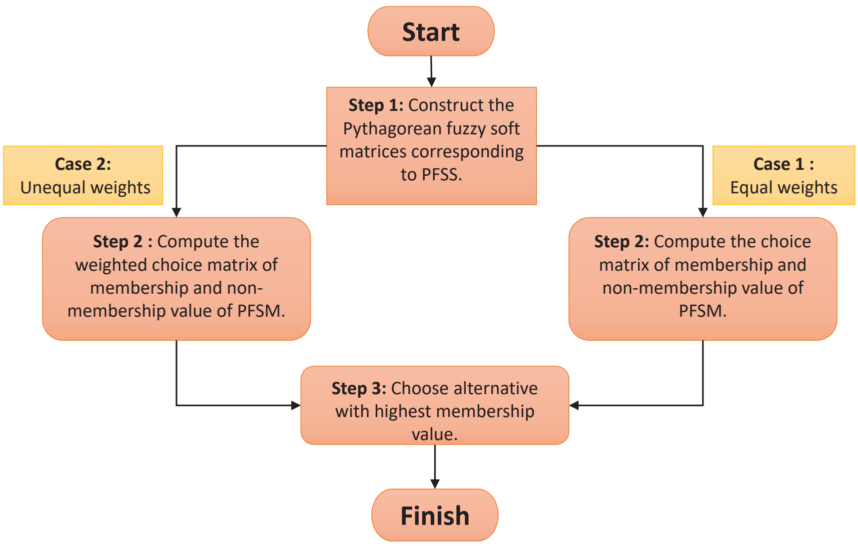

Figure 2.1: Flow Chart of the Algorithm for Decision Making

Example 2.1 *Consider an automobile company which produces three types of car c_1, c_2, c_3, i.e., $U = \{c_1, c_2, c_3\}$. Let $E = \{e_1, e_2, e_3\}$ be a set of criteria representing, good mileage (e_1), comfort (e_2), good power steering (e_3) on the basis of which a customer has to decide which car* *to be purchased. Then the above problem can be model by considering the Pythagorean fuzzy soft set (G, E) over U, where G is mapping $G : E \to \mathcal{P}(U)$ which provides the description of car on the basis of different criteria.*

- **Step 1:** Construct the Pythagorean fuzzy soft matrix:

$$A = \begin{array}{cccc} & e_1 & e_2 & e_3 \\ c_1 & (0.8, 0.5) & (0.6, 0.6) & (0.8, 0.2) \\ c_2 & (0.6, 0.5) & (0.7, 0.4) & (0.8, 0.4) \\ c_3 & (0.5, 0.7) & (0.7, 0.6) & (0.9, 0.3) \end{array}$$

- **Step 2:**
  - **Case 1: Equal weights**

    Evaluate the choice matrix for the Pythagorean fuzzy soft matrix A as :

$$C(\ A\ ) = \begin{bmatrix} (0.5467, 0.2167) \\ (0.4967, 0.19) \\ (0.5167, 0.3133) \end{bmatrix}$$

  - **Case 2: Unequal weights**

    If the weights $0.2, 0.6, 0.2$ are given for the parameters good mileage, comfort, good power steering respectively then the weighted choice matrix for A is as

$$C_w(\ A\ ) = \begin{bmatrix} (0.472, 0.274) \\ (0.494, 0.178) \\ (0.506, 0.332) \end{bmatrix}$$

- **Step 3:**
  - **Case 1 (Equal weights):** From the matrix obtained in Step 2, it is clear that if we give equal preference for all the parameters, we have 0.5467 as the highest membership value, i.e., of car c_1. Therefore, in this case the most suitable car for the customer would be c_1.

  - **Case 2 (Unequal weights):** However, it may also be observed that if the customer gives preference for the parameter "*comfort*" over the other parameters, then 0.506 being the highest membership value for car c_3. Therefore, in this case the most suitable car for the customer would be c_3.

2.3 PFSM in Medical diagnosis

By reframing the definitions of score/utility matrix, an algorithm (Figure 2.2) to solve the medical diagnosis problem has been provided in this section.

Definition 2.3.1 *If* $A = [(\mu^A_{ij}, \nu^A_{ij})] \in PFSM_{m\times n}$, *then the score matrix of Pythagorean fuzzy soft matrix* A *is given by* $S(A) = [s_{ij}] = [((\mu^A_{ij})^2 - (\nu^A_{ij})^2)]$ *for all* i *and* j. *In literature, the* $(i, j)^{th}$ *entry of the score matrix is considered to be an important index for measuring the optimized magnitude of the belongingness/non-belongingness of* i^{th} *patient having a chance of* j^{th} *disease.*

Definition 2.3.2 *If* $A = [(\mu^A_{ij}, \nu^A_{ij})], B = [(\mu^B_{ij}, \nu^B_{ij})] \in PFSM_{m\times n}$ *then the utility matrix of Pythagorean fuzzy soft matrices* A *and* B *is given by* $U(A, B) = [u_{ij}]_{m\times n} = [S(A) - S(B)]$ $\forall$ i*and* j. *It may also be noted that the* $(i, j)^{th}$ *entry of the utility matrix represents another important index for measuring the mixed magnitude of the belongingness in connection with its non-belongingness of* i^{th} *patient having a chance of* j^{th} *disease.*

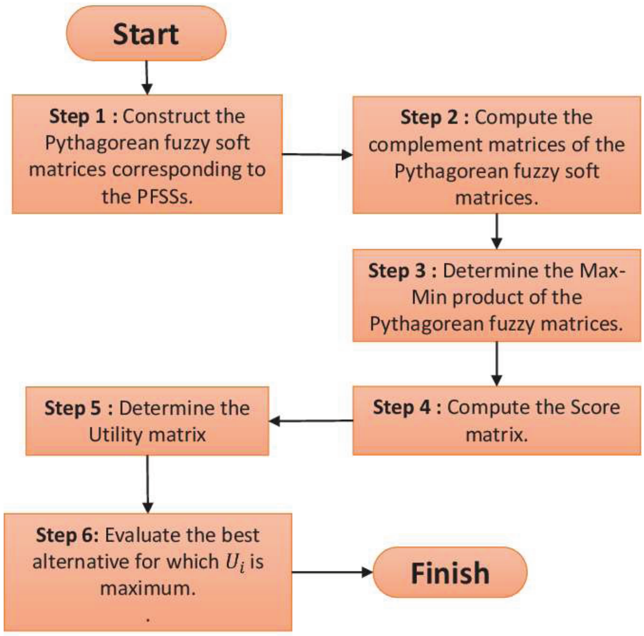

Figure 2.2: Flow Chart of the Algorithm for Medical Diagnosis

In order to break the tie in the repeating values obtained in Step 6, we have to reassess the characteristic values for symptoms and proceed from Step 1 to Step 6 again.

Now, to demonstrate the process of algorithm, the methodology has been presented with the help of numerical example as follows:

Example 2.2 *[40]"Suppose a doctor wants to make a proper diagnosis $D = \{d_1, d_2, d_3, d_4, d_5\}$; where d_1 is Viral fever, d_2 is Malaria, d_3 is Typhoid, d_4 is Stomach problem and d_5 is Chest problem, for a set of patients $P = \{Ted, Al, Bob, Joe\}$ with the values of symptoms $V = \{v_1, v_2, v_3, v_4, v_5\}$; where v_1 is temperature, v_2 is headache, v_3 is Stomach pain, v_4 is cough and v_5 is chest pain."*

- **Step 1:** To understand the problem mathematically, we consider PFSS (F, V) over P, where F is mapping $F : V \to \mathcal{P}(P)$ which represents the description of patient's symptoms in the hospital.

  Step 1:

$$(F, V) = \left\{ \begin{array}{l} F(v_1) = \{(Al, 0.8, 0.1), (Bob, 0.0, 0.8), (Joe, 0.8, 0.1), (Ted, 0.6, 0.1)\} \\ F(v_2) = \{(Al, 0.6, 0.1), (Bob, 0.4, 0.4), (Joe, 0.8, 0.1), (Ted, 0.5, 0.4)\} \\ F(v_3) = \{(Al, 0.2, 0.8), (Bob, 0.6, 0.1), (Joe, 0.0, 0.6), (Ted, 0.3, 0.4)\} \\ F(v_4) = \{(Al, 0.6, 0.1), (Bob, 0.1, 0.7), (Joe, 0.2, 0.7), (Ted, 0.7, 0.2)\} \\ F(v_5) = \{(Al, 0.1, 0.6), (Bob, 0.1, 0.8), (Joe, 0.0, 0.5), (Ted, 0.3, 0.4)\} \end{array} \right\}$$

Further, we transform the PFSS to following PFSM as follows:

$$M = \begin{array}{cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ Al & (0.8, 0.1) & (0.6, 0.1) & (0.2, 0.8) & (0.6, 0.1) & (0.1, 0.6) \\ Bob & (0.0, 0.8) & (0.4, 0.4) & (0.6, 0.1) & (0.1, 0.7) & (0.1, 0.8) \\ Joe & (0.8, 0.1) & (0.8, 0.1) & (0.0, 0.6) & (0.2, 0.7) & (0.0, 0.5) \\ Ted & (0.6, 0.1) & (0.5, 0.4) & (0.3, 0.4) & (0.7, 0.2) & (0.3, 0.4) \end{array}$$

Now, we take the PFSS (G, D) over V, where $G : D \to \mathcal{P}(V)$.

$$(G, D) = \left\{ \begin{array}{l} G(d_1) = \{(v_1, 0.4, 0.0), (v_2, 0.3, 0.5), (v_3, 0.1, 0.7), (v_4, 0.4, 0.3), (v_5, 0.1, 0.7)\} \\ G(d_2) = \{(v_1, 0.7, 0.0), (v_2, 0.2, 0.6), (v_3, 0.0, 0.9), (v_4, 0.7, 0.0), (v_5, 0.1, 0.8)\} \\ G(d_3) = \{(v_1, 0.3, 0.3), (v_2, 0.6, 0.1), (v_3, 0.2, 0.7), (v_4, 0.2, 0.6), (v_5, 0.1, 0.9)\} \\ G(d_4) = \{(v_1, 0.1, 0.7), (v_2, 0.2, 0.4), (v_3, 0.8, 0.0), (v_4, 0.2, 0.7), (v_5, 0.2, 0.7)\} \\ G(d_5) = \{(v_1, 0.1, 0.8), (v_2, 0.0, 0.8), (v_3, 0.2, 0.8), (v_4, 0.2, 0.8), (v_5, 0.8, 0.1)\} \end{array} \right\}$$

Next, we construct the PFSM N as follows:

$$N = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ v_1 & (0.4,0.0) & (0.7,0.0) & (0.3,0.3) & (0.1,0.7) & (0.1,0.8) \\ v_2 & (0.3,0.5) & (0.2,0.6) & (0.6,0.1) & (0.2,0.4) & (0.0,0.8) \\ v_3 & (0.1,0.7) & (0.0,0.9) & (0.2,0.7) & (0.8,0.0) & (0.2,0.8) \\ v_4 & (0.4,0.3) & (0.7,0.0) & (0.2,0.6) & (0.2,0.7) & (0.2,0.8) \\ v_5 & (0.1,0.7) & (0.1,0.8) & (0.1,0.9) & (0.2,0.7) & (0.8,0.1) \end{array}$$

- **Step 2:** In this step, we evaluate the complement matrices corresponding to the PFSMs M and N as follows:

$$M^c = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ Al & (0.1,0.8) & (0.1,0.6) & (0.8,0.2) & (0.1,0.6) & (0.6,0.1) \\ Bob & (0.8,0.0) & (0.4,0.4) & (0.1,0.6) & (0.7,0.1) & (0.8,0.1) \\ Joe & (0.1,0.8) & (0.1,0.8) & (0.6,0.0) & (0.7,0.2) & (0.5,0.0) \\ Ted & (0.1,0.6) & (0.4,0.5) & (0.4,0.3) & (0.2,0.7) & (0.4,0.3) \end{array}$$

$$N^c = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ v_1 & (0.0,0.4) & (0.0,0.7) & (0.3,0.3) & (0.7,0.1) & (0.8,0.1) \\ v_2 & (0.5,0.3) & (0.6,0.2) & (0.1,0.6) & (0.4,0.2) & (0.8,0.0) \\ v_3 & (0.7,0.1) & (0.9,0.0) & (0.7,0.2) & (0.0,0.8) & (0.8,0.2) \\ v_4 & (0.3,0.4) & (0.0,0.7) & (0.6,0.2) & (0.7,0.2) & (0.8,0.2) \\ v_5 & (0.7,0.1) & (0.8,0.1) & (0.9,0.1) & (0.7,0.2) & (0.1,0.8) \end{array}$$

- **Step 3:** In this step, we find the max-min products of the obtained PFSMs.

$$R_1 = M * N = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ Al & (0.4,0.1) & (0.7,0.1) & (0.6,0.1) & (0.2,0.4) & (0.2,0.6) \\ Bob & (0.3,0.5) & (0.4,0.6) & (0.4,0.4) & (0.6,0.1) & (0.2,0.8) \\ Joe & (0.4,0.1) & (0.6,0.1) & (0.7,0.1) & (0.2,0.4) & (0.2,0.5) \\ Ted & (0.7,0.1) & (0.7,0.1) & (0.5,0.3) & (0.3,0.4) & (0.3,0.4) \end{array}$$

$$R_2 = M^c * N^c = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ Al & (0.7,0.1) & (0.7,0.2) & (0.7,0.1) & (0.6,0.2) & (0.8,0.2) \\ Bob & (0.7,0.1) & (0.8,0.1) & (0.8,0.1) & (0.7,0.1) & (0.8,0.1) \\ Joe & (0.6,0.1) & (0.6,0.1) & (0.6,0.1) & (0.7,0.2) & (0.7,0.2) \\ Ted & (0.4,0.3) & (0.6,0.3) & (0.4,0.3) & (0.4,0.3) & (0.4,0.3) \end{array}$$

- **Step 4:** Next, we compute the score matrices for the PFSMs R_1 and R_2 as follows:

$$S(R_1) = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline Al & 0.15 & 0.48 & 0.35 & -0.12 & -0.32 \\ Bob & -0.16 & -0.20 & 0.0 & 0.35 & -0.60 \\ Joe & 0.15 & .35 & 0.48 & -0.12 & -0.21 \\ Ted & 0.48 & 0.48 & 0.16 & -0.07 & -0.07 \end{array}$$

$$S(R_2) = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline Al & .48 & 0.45 & 0.48 & 0.32 & 0.60 \\ Bob & 0.48 & 0.63 & 0.63 & 0.48 & 0.63 \\ Joe & 0.35 & 0.35 & 0.35 & 0.45 & 0.45 \\ Ted & 0.07 & 0.27 & 0.07 & 0.07 & 0.07 \end{array}$$

- **Step 5:** In this step, we find the utility matrix of $S(R_1)$ & $S(R_2)$.

$$U = \begin{array}{c|ccccc} & d_1 & d_2 & d_3 & d_4 & d_5 \\ \hline Al & -0.33 & 0.03 & -0.13 & -0.44 & -0.92 \\ Bob & -0.64 & -0.83 & -0.63 & -0.13 & -1.23 \\ Joe & -0.20 & 0.0 & 0.13 & -0.57 & -0.66 \\ Ted & 0.41 & 0.21 & 0.09 & -0.14 & -0.14 \end{array}$$

- **Step 6:** By observing the entries of the utility matrix obtained above, it probably appears that Al is suffering from Malaria(d_2), Bob is suffering from Stomach problem (d_4), Joe is suffering from Typhoid (d_3) & Ted is suffering form Viral fever (d_1).

Observations : In order to carry out a valid comparative study, we have compared the results obtained by the proposed methodology with the results of various existing methodologies for the same diagnosis problem.

	Al	*Bob*	*Joe*	*Ted*
S^Y [80]	Viral fever	Stomach Problem	*Typhoid*	Viral fever
Szmidt & Kacprzyk[48]	Viral fever	Stomach Problem	*Typhoid*	Malaria
S_{CC} [140]	*Malaria*	Stomach Problem	*Typhoid*	*Malaria*
Wei et al.[28]	*Malaria*	Stomach Problem	*Typhoid*	Viral fever
$p = 1 \quad S_M$ [62]	*Malaria*	Stomach Problem	*Typhoid*	Viral fever
S_1 [171]	*Malaria*	Stomach Problem	*Typhoid*	Viral fever
Proposed Algorithm	*Malaria*	Stomach Problem	*Typhoid*	Viral fever

Figure 2.3: Comparative study w.r.t Existing Methodologies

2.4 Conclusion

The concept of the Pythagorean fuzzy soft matrix has been well established along with its various types and properties. Valid proofs for the proposed properties over the matrices have also been provided. Further, the proposed algorithms for decision making by using choice matrix and weighted choice matrix and for medical diagnosis problem by using score and utility matrix have been successfully implemented with the help of numerical example for each. Further, the comparative analysis shows that the results of the proposed methodology is equally consistent with the results of various other existing methods available in literature.

Chapter 3

Pythagorean Fuzzy Decision Making With Dimensionality Reduction

Dimensionality reduction is a methodology that set out to broaden an arrangement of set of high dimensional data to a lower dimensional data while acquiring the important feature in the data. Because of the inherited disadvantage of dimensionality, the machine learning and data mining techniques may not be successful for high dimensional data. There are two noteworthy techniques for dimensionality reduction - *feature selection* and *feature extraction/feature reduction.*

The problem of dimensionality reduction by utilizing the notion of Pythagorean fuzzy soft matrix (PFSM) has not been addressed yet. In this chapter, in order to handle the parametrization tool in a more effective way, we have suitably extended the literature for reducing the dimensionality of data and compared with the existing methodologies. The definition of the object-oriented PFSM, the parameter-oriented PFSM and the technique to find the threshold element and corresponding threshold value of the PFSM have also been presented in order to propose the algorithm for the dimensionality reduction of the informational data. The comparative analysis along with the advantages of the proposed algorithm has also been presented with the help of numerical examples.

3.1 Algorithm for Dimensionality Reduction

In this section, we propose a new algorithm for the dimensionality reduction of informational data along with the definitions of *object-oriented Pythagorean fuzzy soft matrix* and *parameter-oriented Pythagorean fuzzy soft matrix.*

In general, consider $X = \{x_1, x_2, ..., x_m\}$ be the universe of discourse with the set of parameters $E = \{e_1, e_2, \ldots, e_n\}$ and M be the PFSM of the PFSS (F, E).

Definition 3.1.1 *The object-oriented Pythagorean fuzzy soft matrix with respect to the parameters is defined as:*

$$O_i = \left[\sum_j \frac{\mu_{ij}}{|E|}, \sum_j \frac{\nu_{ij}}{|E|} \right]; \tag{3.1.1}$$

where, $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$.

Definition 3.1.2 *The parameter-oriented Pythagorean fuzzy soft matrix with respect to the objects is defined as:*

$$P_j = \left[\sum_j \frac{\mu_{ij}}{|X|}, \sum_j \frac{\nu_{ij}}{|X|} \right]; \tag{3.1.2}$$

where, $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$.

Definition 3.1.3 *[6] If* $M = [(\mu_{ij}^M, \nu_{ij}^M)] \in PFSM_{m\times n}$*, then the score matrix of Pythagorean fuzzy soft matrix* M *is given by*

$$S(M) = [s_{ij}] = [((\mu_{ij}^M)^2 - (\nu_{ij}^M)^2)] \ \ \forall\, i \text{ and } j; \tag{3.1.3}$$

where, $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$.

Definition 3.1.4 *The threshold value of Pythagorean fuzzy soft matrix is defined as* $S(T) = (\mu_{ij}^M)^2 - (\nu_{ij}^M)^2$*, where*

$$T = (\mu_T, \nu_T) = \left[\sum_{i,j} \frac{\mu_{ij}}{|X \times E|}, \sum_{i,j} \frac{\nu_{ij}}{|X \times E|} \right]; \tag{3.1.4}$$

and $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$.

In view of above definitions and by taking the idea of PFSM into account, an algorithm for the dimensionality reduction of data has been provided in the Figure 3.1.

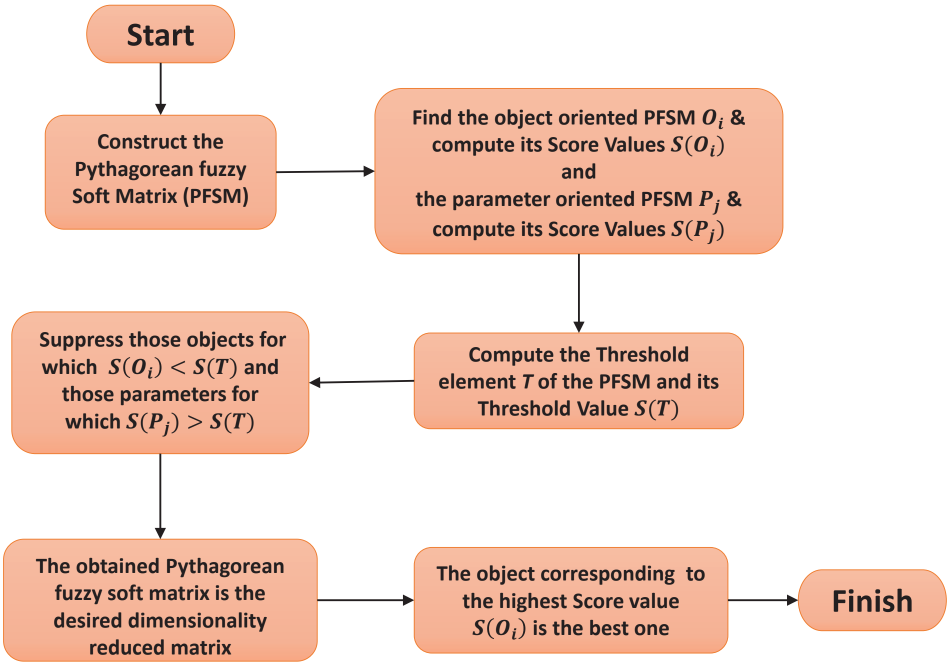

Figure 3.1: Flow Chart of Algorithm for Dimensionality Reduction

3.2 Application in Decision Making

For the better understanding of the proposed algorithm, the step by step implementation of the methodology has been present with the help of numerical example.

Example 3.1 *Let us assume that a person wants to buy a house from the set of houses* $X = \{x_1, x_2, x_3, x_4, x_5\}$ *and the parameter under consideration are* $E = \{e_1, e_2, e_3, e_4\}$ *where,* e_1 : *expensive house,* e_2 : *modern beautiful house,* e_3 : *wooden house in green surrounding,* e_4 : *cheap in bad repair house. Then the attractiveness of the house is described by the Pythagorean fuzzy soft set*

$$(F, E) = \{F(e_1), F(e_2), F(e_3), F(e_4)\} \text{ where } F : E \to PFS(X)$$

and

$$F(e_1) = \{(x_1, 0.7, 0.2), (x_2, 0.9, 0.1), (x_3, 0.4, 0.8), (x_4, 0.3, 0.7), (x_5, 0.8, 0.2)\}$$
$$F(e_2) = \{(x_1, 0.5, 0.6), (x_2, 0.2, 0.6), (x_3, 0.6, 0.5), (x_4, 0.5, 0.5), (x_5, 0.9, 0.1)\}$$
$$F(e_3) = \{(x_1, 0.6, 0.4), (x_2, 0.3, 0.8), (x_3, 0.7, 0.3), (x_4, 0.9, 0.1), (x_5, 0.6, 0.6)\}$$
$$F(e_4) = \{(x_1, 0.4, 0.3), (x_2, 0.8, 0.4), (x_3, 0.7, 0.4), (x_4, 0.9, 0.2), (x_5, 0.7, 0.5)\}$$

For implementing the proposed algorithm taking the above problem into consideration, the computational steps are as follows:

- **Step 1.** Construct the PFSM as follows:

$$M = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} (0.7,0.2) & (0.5,0.6) & (0.6,0.4) & (0.4,0.3) \\ (0.9,0.1) & (0.2,0.6) & (0.3,0.8) & (0.8,0.4) \\ (0.4,0.8) & (0.6,0.5) & (0.7,0.3) & (0.7,0.4) \\ (0.3,0.7) & (0.5,0.5) & (0.9,0.1) & (0.9,0.2) \\ (0.8,0.2) & (0.9,0.1) & (0.6,0.6) & (0.7,0.5) \end{pmatrix} \end{array}$$

- **Step 2.** Evaluate the object oriented PFSM O_i for $i = 1, \dots, 5$ and parameter oriented PFSM P_j for $j = 1, \dots, 4$.

$$M = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ P_j \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & O_i \end{array} \\ \begin{pmatrix} (0.7,0.2) & (0.5,0.6) & (0.6,0.4) & (0.4,0.3) & \mathbf{(0.55,0.375)} \\ (0.9,0.1) & (0.2,0.6) & (0.3,0.8) & (0.8,0.4) & \mathbf{(0.55,0.475)} \\ (0.4,0.8) & (0.6,0.5) & (0.7,0.3) & (0.7,0.4) & \mathbf{(0.60,0.500)} \\ (0.3,0.7) & (0.5,0.5) & (0.9,0.1) & (0.9,0.2) & \mathbf{(0.65,0.375)} \\ (0.8,0.2) & (0.9,0.1) & (0.6,0.6) & (0.7,0.5) & \mathbf{(0.75,0.350)} \\ \mathbf{(0.62,0.40)} & \mathbf{(0.54,0.46)} & \mathbf{(0.62,0.44)} & \mathbf{(0.70,0.36)} & \end{pmatrix} \end{array}$$

Next, the score matrix of objected oriented matrix $S(O_i)$ and parameter oriented matrix $S(P_j)$ is given as:

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ P_j \\ S(P_j) \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & O_i & S(O_i) \end{array} \\ \begin{pmatrix} (0.7,0.2) & (0.5,0.6) & (0.6,0.4) & (0.4,0.3) & (0.55,0.375) & \mathbf{0.161875} \\ (0.9,0.1) & (0.2,0.6) & (0.3,0.8) & (0.8,0.4) & (0.55,0.475) & \mathbf{0.076875} \\ (0.4,0.8) & (0.6,0.5) & (0.7,0.3) & (0.7,0.4) & (0.60,0.500) & \mathbf{0.11} \\ (0.3,0.7) & (0.5,0.5) & (0.9,0.1) & (0.9,0.2) & (0.65,0.375) & \mathbf{0.281875} \\ (0.8,0.2) & (0.9,0.1) & (0.6,0.6) & (0.7,0.5) & (0.75,0.350) & \mathbf{0.44} \\ (0.62,0.40) & (0.54,0.46) & (0.62,0.44) & (0.70,0.36) & & \\ \mathbf{0.2244} & \mathbf{0.08} & \mathbf{0.1908} & \mathbf{0.3604} & & \end{pmatrix} \end{array}$$

- **Step 3.** In this step, we determine the threshold element and threshold value of the PFSM obtained in Step 1 as:

$$T = \Big[(0.62, 0.415)\Big] \text{ and } S(T) = 0.212175.$$

- **Step 4.** Next, we suppress/remove those alternatives and parameters for which the condition $S(O_i) < S(T)$ and $S(P_j) > S(T)$ holds respectively.

  Hence, after suppressing the alternatives and parameters, we obtained our reduced matrix $M^{'}$ as follows:

$$M^{'} = \begin{array}{c} \\ x_4 \\ x_5 \\ S(P_j) \end{array} \begin{array}{c} \begin{array}{ccc} e_2 & e_3 & S(O_i) \end{array} \\ \begin{pmatrix} (0.5, 0.5) & (0.9, 0.1) & 0.281875 \\ (0.9, 0.1) & (0.6, 0.6) & \mathbf{0.44} \\ 0.08 & 0.1908 & \end{pmatrix} \end{array}$$

  In the above reduced matrix, the score value for house x_5 is greater than the score value of the house x_4. Thus, the person will choose the house x_5.

3.3 Comparative Analysis and Advantages

In this section, we carry out a comparative analysis to validate the performance of the proposed methodology in contrast with an existing approach. The detailed analysis and advantages of using the proposed approach along with illustrative example [140] are presented below:

Example: Consider 5 stock sets with high price -earning ratio given by $U = \{x_1, x_2, x_3, x_4, x_5\}$ and 4 sets of evaluation criteria given by $A = \{e_1, e_2, e_3, e_4\}$, where e_1 : market trend, e_2 : policy orientation, e_3 : annual report performance, e_4 : circulation market value. The available data in the form of PFSS presented as follows:

	e_1	e_2	e_3	e_4
x_1	$(0.5, 0.7)$	$(0.6, 0.6)$	$(0.5, 0.6)$	$(0.4, 0.7)$
x_2	$(0.6, 0.6)$	$(0.6, 0.4)$	$(0.7, 0.5)$	$(0.8, 0.4)$
x_3	$(0.8, 0.6)$	$(0.8, 0.3)$	$(0.9, 0.2)$	$(0.6, 0.2)$
x_4	$(0.8, 0.4)$	$(0.4, 0.8)$	$(0.7, 0.6)$	$(0.8, 0.5)$
x_5	$(0.7, 0.6)$	$(0.5, 0.6)$	$(0.6, 0.3)$	$(0.4, 0.6)$

The solution based on the methodology outlined by [140] is as follows:
The score value of the each stock is given by $s(p_1) = -0.1265$, $s(p_2) = 0.2052$, $s(p_3) = 0.5763$, $s(p_4) = 0.0375$, $s(p_5) = 0.0945$, where p_i is the aggregated/integrated representative identity corresponding to each x_i.

Obviously, $s(p_3) > s(p_2) > s(p_5) > s(p_4) > s(p_1)$, therefore, the investor will choose stock x_3 for investment.

On the other hand, if we perform our proposed methodology on the same problem, we find the computations as below:

- **Step 1.** Construct the PFSM as:

$$M = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} (0.5,0.7) & (0.6,0.6) & (0.5,0.6) & (0.4,0.7) \\ (0.6,0.6) & (0.6,0.4) & (0.7,0.5) & (0.8,0.4) \\ (0.8,0.6) & (0.8,0.3) & (0.9,0.2) & (0.6,0.2) \\ (0.8,0.4) & (0.4,0.8) & (0.7,0.6) & (0.8,0.5) \\ (0.7,0.6) & (0.5,0.6) & (0.6,0.3) & (0.4,0.6) \end{pmatrix} \end{array}$$

- **Step 2.** Evaluate the object oriented PFSM O_i for $i = 1, \ldots, 5$ and parameter oriented PFSM P_j for $j = 1, \ldots, 4$.

$$M = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ P_j \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & O_i \end{array} \\ \begin{pmatrix} (0.5,0.7) & (0.6,0.6) & (0.5,0.6) & (0.4,0.7) & \mathbf{(0.5,0.65)} \\ (0.6,0.6) & (0.6,0.4) & (0.7,0.5) & (0.8,0.4) & \mathbf{(0.675,0.475)} \\ (0.8,0.6) & (0.8,0.3) & (0.9,0.2) & (0.6,0.2) & \mathbf{(0.775,0.325)} \\ (0.8,0.4) & (0.4,0.8) & (0.7,0.6) & (0.8,0.5) & \mathbf{(0.675,0.575)} \\ (0.7,0.6) & (0.5,0.6) & (0.6,0.3) & (0.4,0.6) & \mathbf{(0.550,0.525)} \\ \mathbf{(0.68,0.58)} & \mathbf{(0.58,0.54)} & \mathbf{(0.68,0.44)} & \mathbf{(0.60,0.48)} & \end{pmatrix} \end{array}$$

Next, the score matrix of objected oriented matrix $S(O_i)$ and parameter oriented matrix $S(P_j)$ is given as:

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ P_j \\ S(P_j) \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & O_i & S(O_i) \end{array} \\ \begin{pmatrix} (0.5,0.7) & (0.6,0.6) & (0.5,0.6) & (0.4,0.7) & (0.5,0.65) & \mathbf{-0.1725} \\ (0.6,0.6) & (0.6,0.4) & (0.7,0.5) & (0.8,0.4) & (0.675,0.475) & \mathbf{0.23} \\ (0.8,0.6) & (0.8,0.3) & (0.9,0.2) & (0.6,0.2) & (0.775,0.325) & \mathbf{0.495} \\ (0.8,0.4) & (0.4,0.8) & (0.7,0.6) & (0.8,0.5) & (0.675,0.575) & \mathbf{0.125} \\ (0.7,0.6) & (0.5,0.6) & (0.6,0.3) & (0.4,0.6) & (0.550,0.525) & \mathbf{0.026875} \\ (0.68,0.58) & (0.58,0.54) & (0.68,0.44) & (0.60,0.48) & & \\ \mathbf{0.126} & \mathbf{0.0448} & \mathbf{0.2688} & \mathbf{0.1296} & & \end{pmatrix} \end{array}$$

- **Step 3.** In this step, we determine the threshold element and threshold value of the PFSM obtained in Step 1 as:

$$T = \Big[(0.635, 0.51)\Big] \text{ and } S(T) = 0.143125.$$

- **Step 4.** Next, we suppress/remove those alternatives and parameters for which the condition $S(O_i) < S(T)$ and $S(P_j) > S(T)$ holds respectively.

  Hence, after suppressing the alternatives and parameters, we obtained our reduced matrix $M^{'}$ as follows:

$$M^{'} = \begin{array}{c} \\ x_2 \\ x_3 \\ S(P_j) \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & S(O_i) \end{array} \\ \begin{pmatrix} (0.6, 0.6) & (0.6, 0.4) & (0.8, 0.4) & 0.23 \\ (0.8, 0.6) & (0.8, 0.3) & (0.6, 0.2) & \mathbf{0.495} \\ 0.126 & 0.448 & 0.1296 & \end{pmatrix} \end{array}$$

  In the above reduced matrix, the score value for stock x_3 is greater than the score value for stock x_2, therefore, the investor will prefer to invest in the stock x_3.

Comparative Remarks and Advantages of Proposed Work:

In the light of above investigation, the significant comparative remarks and advantages of the proposed work are as follows:

- The methodology utilized by Peng et al.[140] to solve the problem of decision-making doesn't incorporate the theory of dimensional reduction, whereas the proposed methodology has first dimensionally reduced the undesirable data and afterward worked out to find the optimal alternative i.e., the stock x_3 is the most suitable choice for investment.

- Thus, the proposed algorithm for dimensionality reduction is found to be equally reliable, consistent, practicable and better enough for solving decision-making problems by using the notion of PFSM in contrast with the methodologies available in the literature.

- The proposed dimensionality reduction technique associate with the theory of matrices and will prove to be widely applicable in other real world application problems.

- The proposed methodology can also be utilized in the case of large informational data set under the framework of PFSMs.

3.4 Conclusion

In this chapter, an algorithm to reduce the dimensionality of the informational data by utilizing the notion of PFSM has been provided successfully along with the reframing of the definitions of object and parameter oriented PFSMs. Also, a new approach to find the threshold element and its corresponding threshold value has been presented. In order to demonstrate the methodology of the proposed technique a numerical example has been taken into account. A valid comparative study has been provided to show the consistency, practicability, reliability and flexibility of the proposed algorithm in contrast with the existing methodology. The obtained results also validate our contribution and advantages of the proposed algorithm which effectively deal with the dimension reduction.

Chapter 4

Parametric Pythagorean Fuzzy Entropy Measure

In this chapter, we propose a new parametric Pythagorean fuzzy (R, S)-norm entropy measure and also devise two methodologies for finding the criteria weights by incorporating the entropy measure. Empirically, we have also studied the maximality feature and monotonicity of the proposed entropy measure w.r.t. the parameters R & S. An algorithm to solve the multi-criteria decision-making problem by utilizing the proposed entropy measure has also been presented for two different cases- criteria weights are unknown; criteria weights are partially known. In order to demonstrate the methodology of the proposed algorithm, each considered case has been dealt separately with the help of numerical examples.

4.1 Parametric (R, S)-norm Entropy Measure

Recently, Joshi and Kumar [102] proposed and studied a real valued probability distribution function associated with the random variable $X = \{x_1, x_{2,}, \ldots, x_n\}$ which is given as:

$$H_R^S(P) = \frac{R \times S}{R - S}\left[\left(\sum_{i=1}^{n} p_i^S\right)^{\frac{1}{S}} - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]; \tag{4.1.1}$$

where $0 < S < 1$ and $1 < R < \infty$, or $0 < S < 1$ and $1 < R < \infty$.

In particular, this measure reduces to the measure presented by Boekee and Lubbe [30] if the value of S=1 or R=1 as well as if we consider the case $R = 1$ and $S \to 1$ or vice-versa then this entropy measure reduces to Shannon's [16] entropy.

In view of the definition of entropy measure given by Hung & Yang [129] under intuitionistic fuzzy set, we reframe the definition of entropy measure for Pythagorean fuzzy set as follows: Let there be a real valued function $H : X \rightarrow [0,\ 1]$. Then H is a Pythagorean fuzzy entropy measure if and only if it satisfies the following axioms:

- **(PFS1) Sharpness :** $H(M) = 0$ iff M is a crisp set, i.e., $\mu_M(x_i) = 0,\ \nu_M(x_i) = 1$; or $\mu_M(x_i) = 1,\ \nu_M(x_i) = 0;\ \forall x_i \in X$.
- **(PFS2) Maximality :** $H(M)$ is maximum iff
$$\mu_M(x_i) = \nu_M(x_i) = \pi_M(x_i) = \frac{1}{\sqrt{3}} \forall x_i \in X.$$
- **(PFS3) Symmetry :** $H(M) = H(M^c)$.
- **(PFS4) Resolution :** $H(M) \leq H(N)$ iff $M \subseteq N$, i.e., $\mu_M(x_i) \leq \mu_N(x_i)$ and $\nu_M(x_i) \geq \nu_N(x_i)$ for $\mu_N(x_i) \leq \nu_N(x_i)$ or if $\mu_M(x_i) \geq \mu_N(x_i)$ and $\nu_M(x_i) \leq \nu_N(x_i)$ for $\mu_N(x_i) \geq \nu_N(x_i)\ \forall\ x_i \in X$.

For consideration of Pythagorean fuzzy information, the following entropy measure (4.1.2) is being proposed:

$$H_R^S(M) = \begin{cases} \frac{R\,\times\,S}{(\,R-\,S)} \sum\limits_{i=1}^{n} \frac{1}{n} \left[\left(\mu_M(x_i)^{2S} + \nu_M(x_i)^{2S} + \pi_M(x_i)^{2S}\right)^{\frac{1}{S}} - \left(\mu_M(x_i)^{2R} + \nu_M(x_i)^{2R} + \pi_M(x_i)^{2R}\right)^{\frac{1}{R}} \right], \\ \text{where } R, S > 0;\ \text{either } 0 < S < 1 \text{ and } 1 < R < \infty \text{ or } 0 < R < 1 \text{ and } 1 < S < \infty, \\ \\ \frac{R}{n(R-1)} \sum\limits_{i=1}^{n} \left\{ 1 - \left(\mu_M(x_i)^{2R} + \nu_M(x_i)^{2R} + \pi_M(x_i)^{2R}\right)^{\frac{1}{R}} \right\}, \text{ where } S = 1,\ R > 0,\ R \neq 1, \\ \\ -\frac{1}{n} \sum\limits_{i=1}^{n} \left(\mu_M(x_i)^2 log\left(\mu_M(x_i)^2\right) + \nu_M(x_i)^2 log\left(\nu_M(x_i)^2\right) + \pi_M(x_i)^2 log\left(\pi_M(x_i)^2\right)\right), \\ \text{where } R = 1 \text{ and } S \rightarrow 1 \text{ or } S = 1 \text{ and } R \rightarrow 1 \end{cases} \tag{4.1.2}$$

Theorem 4.1 *The entropy measure given by equation 4.1.2 is a valid Pythagorean fuzzy information measure.*

Proof : It is sufficient to prove that the axioms PFS1 to PFS4 hold.

- **(PFS1) (Sharpness):** If $H_R^S(M) = 0$, then
$$\left(\mu_M(x_i)^{2S} + \nu_M(x_i)^{2S} + \pi_M(x_i)^{2S}\right)^{\frac{1}{S}} - \left(\mu_M(x_i)^{2R} + \nu_M(x_i)^{2R} + \pi_M(x_i)^{2R}\right)^{\frac{1}{R}} = 0.$$
Since $R,\ S > 0$ $(R \neq 1 \neq S)$, therefore, only following possibility arises:

- Either $\mu_M(x_i) = 1$, i.e., $\nu_M(x_i) = \pi_M(x_i) = 0$,
- $\nu_M(x_i) = 1$ i.e., $\mu_M(x_i) = \pi_M(x_i) = 0$,
- $\pi_M(x_i) = 1$ i.e., $\mu_M(x_i) = \nu_M(x_i) = 0$.

Looking at the above cases we can say that M is a crisp set. Similarly, converse can be proved.

- **(PFS2) (Maximality) :**
In section 4.2, we have empirically proved that $H_R^S(M)$ is maximum iff

$$\mu_M(x_i) = \nu_M(x_i) = \pi_M(x_i) = \frac{1}{\sqrt{3}}.$$

Analytically, we prove the concavity of the $H_R^S(M)$ by calculating its hessian at the critical point, i.e $\frac{1}{\sqrt{3}}$ with particular values of R and S. The Hessian of $H_R^S(M)$ is as $[R > 1\ (= 3)$ and $S < 1\ (= 0.3)]$:

$$H_R^S(M) = \frac{2}{n}\begin{bmatrix} -10.4589 & 2.232816 & 2.232816 \\ 2.232816 & -10.4589 & 2.232816 \\ 2.232816 & 2.232816 & -10.4589 \end{bmatrix}.$$

It may be observed that $H_R^S(M)$ is a negative semi-definite matrix for different possible values of R and S which shows that it is a cancave function. Hence, the concavity of the function establish the maximality property.

- **(PFS3) (Symmetry) :** It is obvious from the definition that

$$H_R^S(M) \ = H_R^S(M^c).$$

- **(PFS4) (Resolution) :** We have

$$\left|\left(\mu_M(x_i) - \frac{1}{\sqrt{3}}\right)\right| + \left|\left(\nu_M(x_i) - \frac{1}{\sqrt{3}}\right)\right| + \left|\left(\pi_M(x_i) - \frac{1}{\sqrt{3}}\right)\right|$$
$$\geq \left|\left(\mu_N(x_i) - \frac{1}{\sqrt{3}}\right)\right| + \left|\left(\nu_N(x_i) - \frac{1}{\sqrt{3}}\right)\right| + \left|\left(\pi_N(x_i) - \frac{1}{\sqrt{3}}\right)\right|;$$

and

$$\left(\mu_M(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \left(\nu_M(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \left(\pi_M(x_i) - \frac{1}{\sqrt{3}}\right)^2$$
$$\geq \left(\mu_N(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \left(\nu_N(x_i) - \frac{1}{\sqrt{3}}\right)^2 + \left(\pi_N(x_i) - \frac{1}{\sqrt{3}}\right)^2;$$

because if $\mu_M(x_i) \leq \mu_N(x_i)$ and $\nu_M(x_i) \leq \nu_N(x_i)$ with max $\{\mu_N(x_i), \nu_N(x_i)\} \leq \frac{1}{\sqrt{3}}$, then $\mu_M(x_i) \leq \mu_N(x_i) \leq \frac{1}{\sqrt{3}}$; $\nu_M(x_i) \leq \nu_N(x_i) \leq \frac{1}{\sqrt{3}}$ and $\pi_M(x_i) \geq \pi_N(x_i) \geq \frac{1}{\sqrt{3}}$ which implies that the above result holds. Similarly, if $\mu_M(x_i) \geq \mu_N(x_i)$ and $\nu_M(x_i) \geq \nu_N(x_i)$ with max $\{\mu_M(x_i), \nu_M(x_i)\} \geq \frac{1}{\sqrt{3}}$, then also the above result holds.

Now, since $H_R^S(M)$ is a concave function on the Pythagorean fuzzy set M, therefore, if $\max\{\mu_M(x_i), \nu_M(x_i)\} \leq \frac{1}{\sqrt{3}}$ then, $\mu_M(x_i) \leq \mu_N(x_i)$ and $\nu_M(x_i) \leq \nu_N(x_i)$ implies $\pi_M(x_i) \geq \pi_N(x_i) \geq \frac{1}{\sqrt{3}}$.
Therefore, by the above explained result, we conclude that $H_R^S(M)$ satisfies condition of resolution PFS4.

Similarly, if $\min\{\mu_M(x_i), \nu_M(x_i)\} \geq \frac{1}{\sqrt{3}}$, then $\mu_M(x_i) \leq \mu_N(x_i)$ and $\nu_M(x_i) \geq \nu_N(x_i)$. By using the above proved result, we conclude that $H_R^S(M)$ satisfies the condition PFS4.

Hence, $H_R^S(M)$ satisfies all the axioms of Pythagorean fuzzy entropy and therefore, $H_R^S(M)$ is a valid measure of Pythagorean fuzzy information.

Theorem 4.2 Suppose M and N are two PFSs over $X = \{x_1, x_2, \ldots, x_n\}$ where $M = \{< x_i, \mu_M(x_i), \nu_M(x_i) >| \ x_i \in X\}$ and $N = \{< x_i, \mu_N(x_i), \nu_N(x_i) >| \ x_i \in X\}$ such that $\forall \ x_i \in X$ either $M \subseteq N$ or $N \subseteq M$. Then

$$H_R^S(M \cup N) + H_R^S(M \cap N) = H_R^S(M) + H_R^S(N).$$

Proof : First, we partition X into two sub-divisions X_1 & X_2 such that
$X_1 = \{x_i \in X \mid M \subseteq N\}$, i.e., $\mu_M(x_i) \leq \mu_N(x_i)$, $\nu_M(x_i) \geq \nu_N(x_i) \ \forall \ x_i \in X_1$;
$X_2 = \{x_i \in X \mid N \subseteq M\}$, i.e., $\mu_M(x_i) \geq \mu_N(x_i)$, $\nu_M(x_i) \leq \nu_N(x_i) \ \forall \ x_i \in X_1$.

Now

$$H_R^S(M \cup N) = \frac{R \times S}{(R-S)} \sum_{i=1}^{n} \frac{1}{n} \left[\begin{array}{c} \left(\mu_{M\cup N}(x_i)^{2S} + \nu_{M\cup N}(x_i)^{2S} + \pi_{M\cup N}(x_i)^{2S}\right)^{\frac{1}{S}} \\ - \left(\mu_{M\cup N}(x_i)^{2R} + \nu_{M\cup N}(x_i)^{2R} + \pi_{M\cup N}(x_i)^{2R}\right)^{\frac{1}{R}} \end{array} \right];$$

which implies

$$\begin{aligned} H_R^S(M \cup N) = {} & \frac{R \times S}{(R-S)} \sum_{X_1} \frac{1}{n} \left[\begin{array}{c} \left(\mu_N(x_i)^{2S} + \nu_N(x_i)^{2S} + \pi_N(x_i)^{2S}\right)^{\frac{1}{S}} \\ - \left(\mu_N(x_i)^{2R} + \nu_N(x_i)^{2R} + \pi_N(x_i)^{2R}\right)^{\frac{1}{R}} \end{array} \right] \\ & + \frac{R \times S}{(R-S)} \sum_{X_2} \frac{1}{n} \left[\begin{array}{c} \left(\mu_M(x_i)^{2S} + \nu_M(x_i)^{2S} + \pi_M(x_i)^{2S}\right)^{\frac{1}{S}} \\ - \left(\mu_M(x_i)^{2R} + \nu_M(x_i)^{2R} + \pi_M(x_i)^{2R}\right)^{\frac{1}{R}} \end{array} \right]. \end{aligned}$$

Similarly,

$$H_R^S(M\cap N) = \frac{R\times S}{(R-S)}\sum_{X_1}\frac{1}{n}\left[\begin{array}{r}\left(\mu_M(x_i)^{2S}+\nu_M(x_i)^{2S}+\pi_M(x_i)^{2S}\right)^{\frac{1}{S}}\\ -\left(\mu_M(x_i)^{2R}+\nu_M(x_i)^{2R}+\pi_M(x_i)^{2R}\right)^{\frac{1}{R}}\end{array}\right]$$
$$+\frac{R\times S}{(R-S)}\sum_{X_2}\frac{1}{n}\left[\begin{array}{r}\left(\mu_N(x_i)^{2S}+\nu_N(x_i)^{2S}+\pi_N(x_i)^{2S}\right)^{\frac{1}{S}}\\ -\left(\mu_N(x_i)^{2R}+\nu_N(x_i)^{2R}+\pi_N(x_i)^{2R}\right)^{\frac{1}{R}}\end{array}\right].$$

On adding the above two terms, we get

$$H_R^S(M\cup N)\ +\ H_R^S(M\cap N) = H_R^S(M)+H_R^S(N).$$

Theorem 4.3

$$H_R^S(M) = H_R^S(M^c) = H_R^S(M\cup M^c) = H_R^S(M\cap M^c).$$

Proof : The proof can easily be carried out.

4.2 Monotonic Nature of Proposed Entropy Measure

The study of maximality and monotonic behaviour of the proposed entropy measure has been carried out in an empirical way. Here, we take four different Pythagorean fuzzy sets M_1, M_2, M_3 and M_4 over the universe of discourse $X = \{x_1\ , x_2\ , x_3\ \}$:

$$M_1 = \{(x_1,\ \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}),(x_2,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}),(x_3,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})\};$$
$$M_2 = \{(x_1,\ 0.6,0.6),(x_2,\ 0.7,0.7),(x_3,\ 0.55,0.55)\};$$
$$M_3 = \{(x_1,\ 0.5,0.6),(x_2,\ 0.2,0.9),(x_3,\ 0.9,0.3)\};$$
$$M_4 = \{(x_1,\ 0.4,0.8),(x_2,\ 0.9,0.4),(x_3,\ 0.7,0.6)\}.$$

Different values of parameters have been taken for detailed study and tabulated the computed values in Table 4.1. On the basis of the tabulated data, the plots are given below in Figure 4.1. It is quite clear that $H_R^S(M)$ takes maximum value when

$$\mu_M(x_i) = \nu_M(x_i) = \pi_M(x_i) = \frac{1}{\sqrt{3}};\ \forall x_i\in X;$$

and is a monotonically decreasing function of R and S.

Table 4.1: Values of Entropy Measure

S	R= 0.15				R = .25				R = 0.4			
	$H_R^S(M_1)$	$H_R^S(M_2)$	$H_R^S(M_3)$	$H_R^S(M_4)$	$H_R^S(M_1)$	$H_R^S(M_2)$	$H_R^S(M_3)$	$H_R^S(M_4)$	$H_R^S(M_1)$	$H_R^S(M_2)$	$H_R^S(M_3)$	$H_R^S(M_4)$
1.2	86.508	74.420	62.939	66.134	8.263	7.241	6.162	6.464	2.618	2.327	1.962	2.065
1.7	83.052	71.448	60.420	63.489	7.727	6.773	5.753	6.038	2.385	2.122	1.771	1.868
2.5	80.576	69.319	58.613	61.592	7.356	6.449	5.467	5.740	2.228	1.983	1.637	1.730
5	78.100	67.190	56.806	59.692	6.996	6.133	5.188	5.446	2.079	1.850	1.509	1.594
7	77.420	66.604	56.309	59.170	6.899	6.048	5.113	5.366	2.039	1.814	1.476	1.557
10	76.917	66.172	55.942	58.784	6.828	5.985	5.058	5.308	2.010	1.788	1.451	1.530
20	76.339	65.674	55.520	58.340	6.746	5.912	4.995	5.241	1.977	1.756	1.423	1.499
40	76.053	65.427	55.311	58.120	6.706	5.877	4.963	5.208	1.961	1.741	1.409	1.484
50	75.996	65.378	55.270	58.076	6.698	5.869	4.957	5.201	1.958	1.738	1.406	1.481
70	75.931	65.322	55.222	58.026	6.689	5.861	4.950	5.194	1.954	1.734	1.403	1.477
100	75.882	65.280	55.187	57.989	6.682	5.855	4.945	5.188	1.951	1.731	1.401	1.475
200	75.826	65.231	55.145	57.945	6.675	5.848	4.939	5.182	1.948	1.728	1.398	1.472
500	75.792	65.202	55.120	57.919	6.670	5.844	4.935	5.178	1.946	1.726	1.396	1.470
700	75.788	65.198	55.116	57.914	6.652	5.839	4.934	5.177	2.080	1.721	1.396	1.470
S	R= 0.50				R = 0.70				R = 0.95			
	$H_R^S(M_1)$	$H_R^S(M_2)$	$H_R^S(M_3)$	$H_R^S(M_4)$	$H_R^S(M_1)$	$H_R^S(M_2)$	$H_R^S(M_3)$	$H_R^S(M_4)$	$H_R^S(M_1)$	$H_R^S(M_2)$	$H_R^S(M_3)$	$H_R^S(M_4)$
1.2	1.858	1.659	1.378	1.457	1.291	1.158	0.924	0.989	1.034	0.930	0.701	0.764
1.7	1.674	1.496	1.220	1.296	1.149	1.031	0.792	0.856	0.912	0.821	0.579	0.641
2.5	1.552	1.388	1.110	1.182	1.054	0.947	0.699	0.759	0.831	0.750	0.494	0.551
5	1.436	1.284	1.005	1.069	0.965	0.868	0.612	0.662	0.756	0.681	0.417	0.459
7	1.405	1.256	0.978	1.038	0.942	0.846	0.590	0.636	0.736	0.662	0.398	0.435
10	1.383	1.235	0.958	1.016	0.925	0.829	0.574	0.616	0.722	0.646	0.385	0.417
20	1.358	1.210	0.935	0.990	0.906	0.808	0.556	0.595	0.706	0.627	0.369	0.397
40	1.346	1.197	0.924	0.977	0.897	0.797	0.547	0.584	0.698	0.616	0.361	0.388
50	1.343	1.195	0.922	0.975	0.895	0.795	0.545	0.582	0.696	0.614	0.360	0.386
70	1.340	1.192	0.919	0.972	0.893	0.793	0.543	0.580	0.694	0.612	0.358	0.384
100	1.338	1.190	0.917	0.970	0.891	0.791	0.541	0.578	0.693	0.610	0.356	0.382
200	1.336	1.187	0.915	0.968	0.889	0.789	0.539	0.576	0.691	0.608	0.355	0.380
500	1.334	1.186	0.913	0.966	0.888	0.787	0.538	0.575	0.690	0.607	0.354	0.379
700	1.334	1.183	0.913	0.966	0.888	0.771	0.538	0.574	0.689	0.592	0.353	0.379

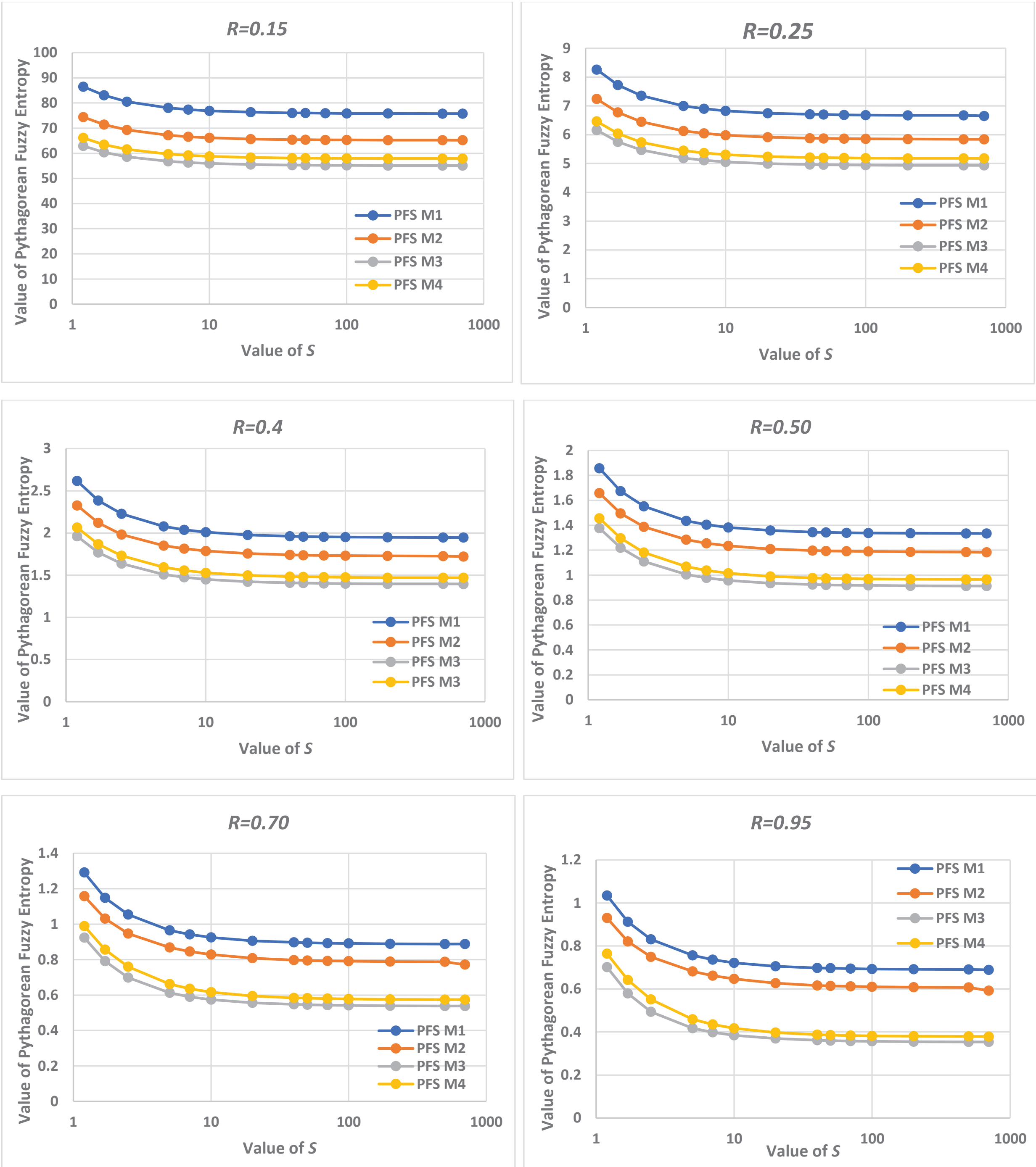

Figure 4.1: Monotonicity of the $(R,\, S)$-norm Entropy Measure

4.3 Decision Making with (R, S)-norm Entropy Measure

The main objective of the multi-criteria decision making problem is to select the optimal/best alternative out of the m feasible available alternatives, i.e., $Z = \{z_1, z_2, \ldots, z_m\}$ based on certain laid down n criteria (say) $O = \{o_1, o_2, \ldots, o_n\}$. For this, first we take the appraisal values of an alternative $z_i\,(i = 1, 2, 3, \ldots, m)$ w.r.t the criteria $o_j\,(j = 1, 2, 3, \ldots, n)$ is given by $z_{ij} = (p_{ij}, q_{ij})$, satisfying $0 \le p_{ij} \le 1$, $0 \le q_{ij} \le 1$ and $0 \le p_{ij} + q_{ij} \le 1$ with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Thus, the above problem can be modeled by representing it through the following Pythagorean fuzzy decision matrix:

$$R = (p_{ij}, q_{ij})_{m\times n} = (z_{ij}) = \begin{array}{cccccc} & o_1 & o_2 & \cdots & o_n \\ z_1 & (p_{11}, q_{11}) & (p_{12}, q_{12}) & \cdots & (p_{1n}, q_{1n}) \\ z_2 & (p_{21}, q_{21}) & (p_{22}, q_{22}) & \cdots & (p_{2n}, q_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_m & (p_{m1}, q_{m1}) & (p_{m2}, q_{m2}) & \cdots & (p_{mn}, q_{mn}) \end{array}$$

Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of all the criteria where $0 \le w_j \le 1$ and $\sum_{j=1}^{n} w_j$ is the degree of importance of the j^{th} criteria. Sometimes this criteria weight is completely unknown and sometimes it is partially known because of the lack of knowledge, time, data and the limited expertise of the problem domain.

In this section, we discuss and devise two methods to determine the weights of criteria by using the proposed entropy (4.1.2).

Case 1 (Unknown Weights) When the criteria weights are completely unknown, then we calculate the weights by using the proposed PFS entropy as:

$$w_j = \frac{1 - e_j}{n - \sum_{j=1}^{n} e_j}, \; j = 1, 2, \cdots, n; \tag{4.3.1}$$

where $e_j = \frac{1}{m} \sum_{i=1}^{m} H_R^S(z_{ij})$, and

$$H_R^S(z_{ij}) = \frac{R \times S}{(R - S)} \sum_{i=1}^{m} \frac{1}{m} \left[\begin{array}{l} \left(\mu_M(x_i)^{2S} + \nu_M(x_i)^{2S} + \pi_M(x_i)^{2S}\right)^{\frac{1}{S}} \\ \quad - \left(\mu_M(x_i)^{2R} + \nu_M(x_i)^{2R} + \pi_M(x_i)^{2R}\right)^{\frac{1}{R}} \end{array} \right]$$

is the proposed Pythagorean fuzzy entropy for $z_{ij} = (p_{ij}, q_{ij})$.

Case 2 (Partially Known Weights) In this case, when the weights are partially known for a multiple-criteria decision making problem, we use the minimum entropy principle (Wang and Wang [64]) to determine the weight vector of the criteria by constructing the programming model as follows:

The overall entropy of the alternative z_i is

$$\begin{aligned}E\left[z_i\right] &= \sum_{j=1}^{n} H_R^S(z_{ij}) \\ &= \frac{R\times S}{(R-S)}\sum_{j=1}^{n}\left\{\sum_{i=1}^{m}\frac{1}{m}\left[\begin{array}{r}\left(\mu_M(x_i)^{2S}+\nu_M(x_i)^{2S}+\pi_M(x_i)^{2S}\right)^{\frac{1}{S}} \\ -\left(\mu_M(x_i)^{2R}+\nu_M(x_i)^{2R}+\pi_M(x_i)^{2R}\right)^{\frac{1}{R}}\end{array}\right]\right\};\end{aligned}$$

where $R,\ S>0$; $R>1,\ S<1$ or $R<1,\ S>1$.

Since there are fair competitive environment between each alternative, the weight coefficient w.r.t the same criteria should also be equal. Further, in order to get the ideal weight, we construct the following accompanying model:

$$\begin{aligned}\min E &= \sum_{i=1}^{m} w_j E\left(z_i\right) = \sum_{i=1}^{m} w_j\left\{\sum_{j=1}^{n} H_R^S(z_{ij})\right\} \qquad (4.3.2) \\ &= \frac{R\ \times\ S}{(\ R-\ S)}\sum_{j=1}^{n} w_j \sum_{i=1}^{m}\frac{1}{m}\left\{\begin{array}{r}\left(\mu_M(x_i)^{2S}+\nu_M(x_i)^{2S}+\pi_M(x_i)^{2S}\right)^{\frac{1}{S}} \\ -\left(\mu_M(x_i)^{2R}+\nu_M(x_i)^{2R}+\pi_M(x_i)^{2R}\right)^{\frac{1}{R}}\end{array}\right\}\end{aligned}$$

$R,\ S>0$; $R>1,\ S<1$ or $R<1,\ S>1$, subject to $\sum_{j=1}^{n}\ w_j=1$.

In view of above two methods and using the notion of PFS, we present an algorithm to solve general MCDM problem as in Figure 4.2.

The procedural steps of the proposed methodology are as follows:

- **Step 1:** We construct the decision matrix $R=(p_{ij},q_{ij})_{m\times n}=o_j(z_i)$, where the elements $o_j(z_i)$ $(i=1,2,...,m;\ j=1,2,...,n)$ are the appraisal of the alternative $z_i\in Z$ w.r.t the criteria $o_j\in O$.
- **Step 2:** Compute the criteria weights by using equation (5.1.1) and (5.1.4).
- **Step 3:** Determine the the most preferred solution (z^+) and the least preferred solution (z^-) as

  $$z^+ \ = \ \left(\left(\alpha_1^+,\beta_1^+\right),\left(\alpha_2^+,\beta_2^+\right),\ldots,\left(\alpha_n^+,\beta_n^+\right)\right);$$

  where $(\alpha_j^+,\beta_j^+)=(\sup\ \mu_M(z_i),\ \inf\ \nu_M(z_i))$, $z_i\in Z$; $(j=1,2,\ldots,n)$; and

  $$z^- = \left(\left(\alpha_1^-,\beta_1^-\right),\left(\alpha_2^-,\beta_2^-\right),\ldots,\left(\alpha_n^-,\beta_n^-\right)\right);$$

  where $(\alpha_j^-,\beta_j^-)=(\inf\ \mu_M(z_i),\ \sup\ \nu_M(z_i))$, $z_i\in Z$; $(j=1,2,\ldots,n)$ respectively.

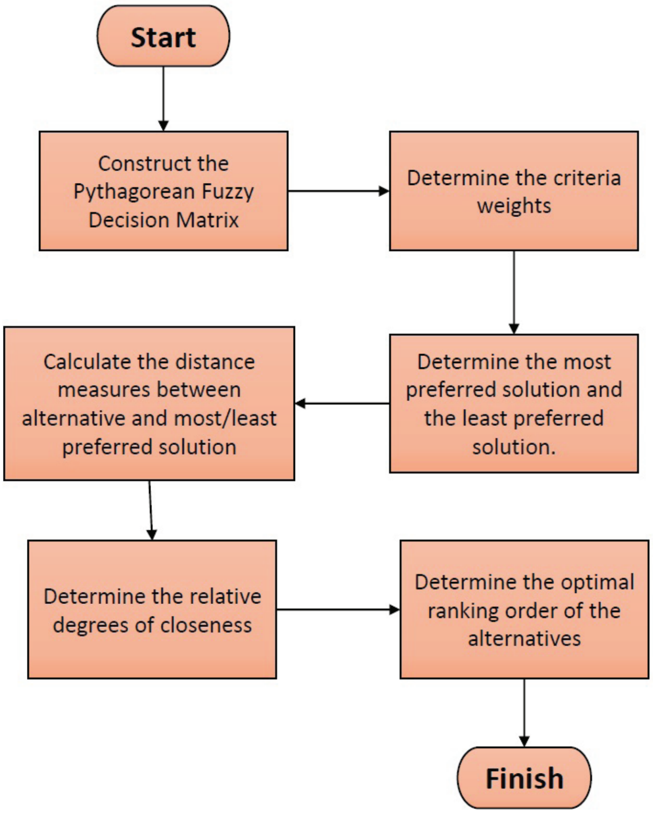

Figure 4.2: Flowchart of the Proposed Algorithm Using PFS

- **Step 4:** By using the Pythagorean fuzzy Hamming distance measure [139]

$$l(M,N)=\frac{1}{2}(|(\mu_M(x))^{2}-(\mu_N(x))^{2}|+|(\nu_M(x))^{2}-(\nu_N(x))^{2}|+|(\pi_M(x))^{2}-(\pi_N(x))^{2}|);$$

we compute the distance of $z_i's$ from z^+ and z^- as follows:

$$l(z_i,z^+)=\frac{1}{2}\sum_{j=1}^{n}w_j\left(|(\alpha_{ij})^{2}-(\alpha_j^+)^{2}|+|(\beta_{ij})^{2}-(\beta_j^+)^{2}|+|(\pi_{ij})^2-(\pi_j^+)^2|\right);$$

and

$$l(z_i,z^-)=\frac{1}{2}\sum_{j=1}^{n}w_j\left(|(\alpha_{ij})^2-(\alpha_j^-)^{2}|+|(\beta_{ij})^{2}-(\beta_j^-)^{2}|+|(\pi_{ij})^2-(\pi_j^-)^2|\right).$$

- **Step 5:** Evaluate the coefficient of degrees of closeness $l_i's$ as :

$$l_i=\frac{l(z_i,z^-)}{l(z_i,z^-)+l(z_i,z^+)}.$$

- **Step 6:** Based on the values obtained in step (5), we determine the optimal ranking order of the alternatives. The alternative with the maximal degree of closeness $l(z_i)$ is supposed to be the best alternative.

4.4 Numerical Examples

In this section, we present two numerical examples on the basis of the considered cases in the proposed algorithm.

Example 1 (Unknown Weights): Assume an automobile company produces 4 different cars, say, z_1, z_2, z_3 & z_4 and a customer wants to buy a car based on the 4 given criteria, say, comfort o_1, good mileage o_2, safety o_3, interiors o_4. Consider the appraisal values of the alternatives w.r.t each criteria provided by the expert is represented as follows:

	o_1	o_2	o_3	o_4
z_1	$(0.9, 0.3)$	$(0.7, 0.6)$	$(0.5, 0.8)$	$0.6, 0.3$
z_2	$(0.4, 0.7)$	$(0.9, 0.2)$	$(0.8, 0.1)$	$(0.5, 0.3)$
z_3	$(0.8, 0.4)$	$(0.7, 0.5)$	$(0.6, 0.2)$	$(0.7, 0.4)$
z_4	$(0.7, 0.2)$	$(0.8, 0.2)$	$(0.8, 0.4)$	$(0.6, 0.6)$

Then, to solve the above problem the computational step are as follows:

1. Determine the criteria weight by using equation (5.1.1) :

$$w = (w_1, w_2, w_3, w_4)^T = (0.272107,\ 0.263037,\ 0.34878,\ 0.116077)^T.$$

2. The most preferred solution (z^+) and the least preferred solution (z^-) are given by

$$z^+ = \{(0.9, 0.3), (0.9, 0.2), (0.8, 0.1), (0.7, 0.4)\}$$

and

$$z^- = \{(0.4, 0.7), (0.7, 0.6), (0.5, 0.8), (0.6, 0.6)\}$$

respectively.

3. The distances measure between each of $z_i's$ from z^+ and z^- are given by

$$l(z_1, z^+) = 0.040622, \quad l(z_2, z^+) = 0.186515, \quad l(z_3, z^+) = 0.0.06623, \quad l(z_4, z^+) = 0.048795,$$
$$l(z_1, z^-) = 0.209804, \quad l(z_2, z^-) = 0.13179, \quad l(z_3, z^-) = 0.177491, \quad l(z_4, z^+) = 0.116968.$$

4. The values of coefficient of degree of closeness are as follows:

$$l_1 = \ 0.837788, \quad l_2 = 0.414036, \quad l_3 = 0.728256 \quad l_4 = 0.705633.$$

5. On the basis of value obtained in above step, the ranking of the alternatives is as follows:

$$z_1 \succ z_3 \succ z_4 \succ z_2$$

and the optimal/best alternative is z_1 among all the available alternative.

Example 2 (Partially Known Weights): Assume there are 1000 students in a college and on the basis of 3 laid down criteria, say, o_1 (personality), o_2 (intelligence) and o_3 (communication skills), the college administration wants to select a college representative. Let there be 3 candidates, say, z_1, z_2 and z_3. The PFS decision matrix for the above problem is

$$\begin{array}{cccc} & o_1 & o_2 & o_3 \\ z_1 & (0.8, 0.5) & (0.6, 0.6) & (0.8, 0.2) \\ z_2 & (0.6, 0.5) & (0.7, 0.4) & (0.8, 0.4) \\ z_3 & (0.5, 0.7) & (0.7, 0.6) & (0.9, 0.3) \end{array}$$

Suppose the partial information about criteria weights is available in the following form $\{0.10 \leq w_1 \leq 0.30,\ 0.35 \leq w_2 \leq 0.60,\ 0.25 \leq w_3 \leq 0.70\}$. The calculation for the ranking procedure for the above decision-making problem is as follows:

1. We calculate the criteria weights by constructing the linear programming model by using equation (5.1.4) as follows:

$$\min E = 0.609037w_1 + 0.641365w_2 + 0.590874w_3$$

subject to $w_1 + w_2 + w_3 = 1$ with possible ranges (careful in taking extremities)

$$\begin{aligned} 0.10 &\leq w_1 \leq 0.30, \\ 0.35 &\leq w_2 \leq 0.60, \\ 0.25 &\leq w_3 \leq 0.70. \end{aligned}$$

Then by using mathematical software MATLAB, we obtained the criteria weight as follows:

$$w = (0.10,\ 0.35,\ 0.55)^T.$$

2. The most preferred solution (z^+) and the least preferred solution (z^-) are given by

$$z^+ = \{(0.8, 0.5), (0.7, 0.4), (0.9, 0.2)\}$$

and

$$z^- = \{(0.5, 0.7), (0.6, 0.6), (0.8, 0.4)\}$$

respectively.

3. The distances measure between each of $z_i's$ from z^+ and z^- are given by

$$l(z_1, z^+) = 0.013843, \quad l(z_2, z^+) = 0.015888, \quad l(z_3, z^+) = 0.068163,$$
$$l(z_1, z^-) = 0.052213, \quad l(z_2, z^-) = 0.026855, \quad l(z_3, z^-) = 0.049273.$$

4. The values of coefficient of degree of closeness are

$$l_1 = \ 0.79044, \quad l_2 = 0.628297, \quad l_3 = 0.419573.$$

5. In view of the values obtained in above step, the ranking of the alternatives is as:

$$z_1 \succ z_2 \succ z_3$$

and the z_1 and is the optimal/best available alternative.

Remark: It may be noted that in the above examples, for the computational procedure we assume the value of $R = 3$ and $S = 0.3$.

4.5 Conclusion

In this chapter, we have successfully proposed a new parametric (R, S)-norm entropy measure for Pythagorean fuzzy set along with the proof of its validity and also studied its maximality and the monotonic behavior w.r.t parameters R & S. Further, an algorithm for multi-criteria decision-making problem has been well proposed and successfully implemented with the help of two different kind of numerical examples- when criteria weights are unknown and other when criteria weights are partially known.

Chapter 5

Pythagorean Fuzzy Parametric Discriminant Measure in Decision Making

In this chapter, we have presented a new parametric Pythagorean fuzzy (R, S)-norm discriminant measure and also discussed its applicability in various computational application fields. Analytically, we have also studied different properties which the proposed discriminant measure holds. We have empirically studied the monotonicity of the proposed measure w.r.t. the parameters R & S. Further, different algorithms to handle the problem of pattern recognition, medical diagnosis and decision making have also been presented and demonstrated with the help of numerical example for each. The comparative remarks in each considered case have been listed depicting the important observations and advantages of the proposed discriminant measure.

5.1 Parametric (R,S)-Norm Discriminant Measure

Recently, Joshi and Kumar [101] proposed and studied a real valued probability distribution function associated with the random variable $X = \{x_1, x_{2,}, \ldots, x_n\}$ and two probability distributions $P = (p_1, p_2, \ldots, p_n)$ and $Q = (q_1, q_2, \ldots, q_n)$ which is given as:

$$D_R^S(P,Q) = \frac{R \times S}{S - R}\left[\left(\sum_{i=1}^{n}(p_i^S q_i^{1-S})\right)^{\frac{1}{S}} - \left(\sum_{i=1}^{n}(p_i^R q_i^{1-S})^{\frac{1}{R}}\right)\right]; \tag{5.1.1}$$

where either $0 < S < 1$ and $1 < R < \infty$ or $0 < R < 1$ and $1 < S < \infty$.

Analogous to the measure in equation (5.1.1), we present the parametric discriminant measure under the Pythagorean fuzzy environment as follows:

$$\begin{aligned} I_R^S(M,N) \;=\; & \frac{R \times S}{n(S-R)} \sum_{i=1}^{n} \Bigg[\Big(\mu_M(x_i)^{2S} \mu_N(x_i)^{2\,(1-S)} + \nu_M(x_i)^{2S} \nu_N(x_i)^{2\,(1-S)} \\ & + \pi_M(x_i)^{2S} \pi_N(x_i)^{2(1-S)} \Big)^{\frac{1}{S}} - \Big(\mu_M(x_i)^{2R} \mu_N(x_i)^{2(1-R)} + \nu_M(x_i)^{2R} \nu_N(x_i)^{2\,(1-R)} \\ & + \pi_M(x_i)^{2R} \pi_N(x_i)^{2\,(1-R)} \Big)^{\frac{1}{R}} \Bigg] \end{aligned} \tag{5.1.2}$$

where either $0 < S < 1$ and $1 < R < \infty$ or $0 < R < 1$ and $1 < S < \infty$.

If $R = 1$ and $S \to 1$ or $S = 1$ and $R \to 1$, then the discriminant measure given by equation (5.1.2) reduces to

$$I(M,N) = \frac{2}{n} \left(\mu_M(x_i)^2 \log\left(\frac{\mu_M(x_i)}{\mu_N(x_i)}\right) + \nu_M(x_i)^2 \log\left(\frac{\nu_M(x_i)}{\nu_N(x_i)}\right) + \pi_M(x_i)^2 \log\left(\frac{\pi_M(x_i)}{\pi_N(x_i)}\right) \right). \tag{5.1.3}$$

It may be noted that proposed discriminant measure is not symmetric in connection with its arguments. Hence, we present the symmetric discriminant measure as follows:

$$J_R^S(M,\ N) \;=\; I_R^S(M,N) + \ I_R^S(N,M). \tag{5.1.4}$$

Under the intuitionistic fuzzy setup, Vlachos and Sergiadis [53] studied the notion of discriminant information measure and defined *intuitionistic fuzzy cross entropy* as $I_{IFS}\ (A,\ B)$ which satisfies two axioms:

- $I_{IFS}\ (A,B)\ \geq 0$;
- $I_{IFS}\ (A,B)\ = 0$ iff $A\ = B$.

Theorem 5.1 *The discriminant measure given by equation (5.1.2) is a valid Pythagorean fuzzy information measure.*

Proof : First, we prove that $I_R^S(M,N) \geq\ 0$ with equality if

$$\mu_M(x_i) \;=\; \mu_N(x_i) \text{ and } \nu_M(x_i) = \ \nu_N(x_i) \text{ for all} i\ = 1,2,\ldots,\ n.$$

Let $\sum_{i=1}^{n} \mu_M(x_i)^2 = a$, $\sum_{i=1}^{n} \mu_N(x_i)^2 = b$, $\sum_{i=1}^{n} \nu_M(x_i)^2 = c$ and $\sum_{i=1}^{n} \nu_N(x_i)^2 = d$, then

$$\sum_{i=1}^{n} \left(\frac{\mu_M(x_i)^{\,2}}{a}\right)^{S} \left(\frac{\mu_N(x_i)^{\,2}}{b}\right)^{(1-S)} \geq 1;$$

or

$$\sum_{i=1}^{n}(\mu_M(x_i)^{2S})(\mu_N(x_i)^{2(1-S)}) \geq a^S b^{1-S}. \tag{5.1.5}$$

Similarly, we have

$$\sum_{i=1}^{n}\left(\frac{\nu_M(x_i)^2}{c}\right)^S \left(\frac{\nu_N(x_i)^2}{d}\right)^{(1-S)} \geq 1;$$

or

$$\sum_{i=1}^{n}(\nu_M(x_i)^{2S})(\nu_N(x_i)^{2(1-S)}) \geq c^S d^{1-S}; \tag{5.1.6}$$

and

$$\sum_{i=1}^{n}(\pi_M(x_i)^{2S})(\pi_N(x_i)^{2(1-S)}) \geq (n-a-b)^S \ (n-c-d)^{1-S}. \tag{5.1.7}$$

From equations (5.1.5), (5.1.6) and (5.1.7), we get

$$\begin{aligned}&\sum_{i=1}^{n}\left(\mu_M(x_i)^{2S}\mu_N(x_i)^{2(1-S)} + \nu_M(x_i)^{2S}\nu_N(x_i)^{2(1-S)} + \pi_M(x_i)^{2S}\pi_N(x_i)^{2(1-S)}\right)\\ &\quad \geq \left(a^S b^{1-S} + c^S d^{1-S} + (n-a-b)^S \ (n-c-d)^{1-S}\right).\end{aligned} \tag{5.1.8}$$

Case 1: $0 < S < 1$ and $1 < R < \infty$.

Let $\mu_M(x_i)^{2S}\mu_N(x_i)^{2(1-S)} + \nu_M(x_i)^{2S}\nu_N(x_i)^{2(1-S)} + \pi_M(x_i)^{2S}\pi_N(x_i)^{2(1-S)} = z_i$. Since $z_i < 1$ and $\frac{1}{S} > 1$, therefore, $z_i > (z_i)^{\frac{1}{S}}$.

As $\frac{R \times S}{n\ (S-R)} < 0$, then

$$\frac{R \ \times S}{n\ (S-R)}\sum_{i=1}^{n}\left[(z_i)^{\frac{1}{S}}\right] > \frac{R \times S}{n(S-R)}\sum_{i=1}^{n}(z_i) \tag{5.1.9}$$

and for $R > 1$,

$$\frac{R \times S}{n(S-R)}\sum_{i=1}^{n}\left[(z_i)^{\frac{1}{R}}\right] < \frac{R \times S}{n(S-R)}\sum_{i=1}^{n}(z_i). \tag{5.1.10}$$

Therefore, from (5.1.9) and (5.1.10), we have $I_R^S(M,N) > 0$ and if $\mu_M(x_i) = \mu_N(x_i)$ and $\nu_M(x_i) = \nu_N(x_i)$ in (5.1.1), we have $I_R^S(M,N) = 0$. Hence, we conclude that $I_R^S(M,N) \geq 0$.

Next we prove the convexity of $I_R^S(M,N)$ in this case.

For $0 < S < 1$, equation (5.1.8) may be written as

$$\begin{aligned}&\left(\sum_{i=1}^{n}\left(\mu_M(x_i)^{2S}\mu_N(x_i)^{2(1-S)} + \nu_M(x_i)^{2S}\nu_N(x_i)^{2(1-S)} + \pi_M(x_i)^{2S}\pi_N(x_i)^{2(1-S)}\right)\right)^{\frac{1}{S}}\\ &\leq \left(a^S b^{1-S} + c^S d^{1-S} + (n-a-b)^S(n-c-d)^{1-S}\right)^{\frac{1}{S}}.\end{aligned}$$

Also, we can write the above equation as

$$\sum_{i=1}^{n}\left[\left(\mu_M(x_i)^{2S}\mu_N(x_i)^{2(1-S)}+\nu_M(x_i)^{2S}\nu_N(x_i)^{2(1-S)}+\pi_M(x_i)^{2S}\pi_N(x_i)^{2(1-S)}\right)^{\frac{1}{S}}\right]$$
$$\leq\left[\sum_{i=1}^{n}\left(\mu_M(x_i)^{2S}\mu_N(x_i)^{2(1-S)}+\nu_M(x_i)^{2S}\nu_N(x_i)^{2(1-S)}+\pi_M(x_i)^{2S}\pi_N(x_i)^{2(1-S)}\right)\right]^{\frac{1}{S}} \tag{5.1.11}$$

Next, for $R>1$, from equation (5.1.8), we have

$$\left(\sum_{i=1}^{n}\left(\mu_M(x_i)^{2R}\mu_N(x_i)^{2(1-R)}+\nu_M(x_i)^{2R}\nu_N(x_i)^{2(1-R)}+\pi_M(x_i)^{2R}\pi_N(x_i)^{2(1-R)}\right)\right)^{\frac{1}{R}}$$
$$\geq\left(a^Rb^{1-R}+c^Rd^{1-R}+(n-a-b)^R(n-c-d)^{1-R}\right)^{\frac{1}{R}};$$

and above equation can be written as

$$\sum_{i=1}^{n}\left[\left(\mu_M(x_i)^{2R}\mu_N(x_i)^{2(1-R)}+\nu_M(x_i)^{2R}\nu_N(x_i)^{2(1-R)}+\pi_M(x_i)^{2R}\pi_N(x_i)^{2(1-R)}\right)^{\frac{1}{R}}\right]$$
$$\geq\left[\sum_{i=1}^{n}\left(\mu_M(x_i)^{2R}\mu_N(x_i)^{2(1-R)}+\nu_M(x_i)^{2R}\nu_N(x_i)^{2(1-R)}+\pi_M(x_i)^{2R}\pi_N(x_i)^{2(1-R)}\right)\right]^{\frac{1}{R}} \tag{5.1.12}$$

Since $\frac{R\times S}{n(S-R)}<0$, therefore, from (5.1.11) and (5.1.12), we get

$$I_R^S(M,N)\geq\frac{R\times S}{n(S-R)}\left[\left(a^Sb^{1-S}+c^Sd^{1-S}+(n-a-b)^S(n-c-d)^{1-S}\right)^{\frac{1}{S}}\right.$$
$$\left.-\left(a^Rb^{1-R}+c^Rd^{1-R}+(n-a-b)^R(n-c-d)^{1-R}\right)^{\frac{1}{R}}\right]. \tag{5.1.13}$$

Further, if we take

$$\phi(a,b)=\frac{R\times S}{n(S-R)}\left[\left(a^Sb^{1-S}+c^Sd^{1-S}+(n-a-b)^S(n-c-d)^{1-S}\right)\right.$$
$$\left.-\left(a^Rb^{1-R}+c^Rd^{1-R}+(n-a-b)^R(n-c-d)^{1-R}\right)\right],$$

then

$$\frac{\partial\phi(a,b)}{\partial a}=\frac{R\times S}{n(S-R)}\left[\left(S\left(\frac{a}{b}\right)^{S-1}-S\left(\frac{n-a-c}{n-b-d}\right)^{S-1}\right)-\right.$$
$$\left.\left(R(\frac{a}{b})^{R-1}-R(\frac{n-a-c}{n-b-d})^{R-1}\right)\right], \tag{5.1.14}$$

and

$$\frac{\partial^2\phi(a,b)}{\partial a^2} = \frac{R\times S}{n(S-R)}\Bigg[\Bigg(\frac{S(S-1)}{b}\left(\frac{a}{b}\right)^{S-2} + \frac{S(S-1)}{n-b-d}\left(\frac{n-a-c}{n-b-d}\right)^{S-2}\Bigg)$$
$$-\Bigg(\frac{R(R-1)}{b}\left(\frac{a}{b}\right)^{R-2} + \frac{R(R-1)}{n-b-d}\left(\frac{n-a-c}{n-b-d}\right)^{R-2}\Bigg)\Bigg] > 0. \tag{5.1.15}$$

Thus, it may be noted that $\phi(a,b)$ is a convex function in a, where the minimum value is attained to be zero when $\frac{a}{b} = \frac{n-a-c}{n-b-d}$.

Hence, $\phi(a,b)$ vanishes only when $a=b$ and $c=d$.

Case 2: $S>1$ and $0<R<1$.

Let $\mu_M(x_i)^{2S}\mu_N(x_i)^{2(1-S)} + \nu_M(x_i)^{2S}\nu_N(x_i)^{2(1-S)} + \pi_M(x_i)^{2S}\pi_N(x_i)^{2(1-S)} = z_i$. Since $z_i<1$ and $\frac{1}{S}<1$, therefore, $z_i < (z_i)^{\frac{1}{S}}$.

As $\frac{R\times S}{n(S-R)} > 0$, therefore,

$$\frac{R\times S}{n(S-R)}\sum_{i=1}^{n}\left[(z_i)^{\frac{1}{S}}\right] > \frac{R\times S}{n(S-R)}\sum_{i=1}^{n}(z_i); \tag{5.1.16}$$

and for $0<R<1$,

$$\frac{R\times S}{n(S-R)}\sum_{i=1}^{n}\left[(z_i)^{\frac{1}{R}}\right] > \frac{R\times S}{n(S-R)}\sum_{i=1}^{n}(z_i). \tag{5.1.17}$$

Therefore, from (5.1.16) and (5.1.17), we have $I_R^S(M,N)>0$ and if $\mu_M(x_i)=\mu_N(x_i)$ and $\nu_M(x_i)=\nu_N(x_i)$ in (5.1.2), we have $I_R^S(M,N)=0$. Hence, we conclude that $I_R^S(M,N)\geq 0$. Further, on similar lines as in case 1, we prove the convexity of $I_R^S(M,N)$ in this case. Consequently, this implies that $I_R^S(M,N)\geq 0$, where equality holds only when $\mu_M(x_i) = \mu_N(x_i), \nu_M(x_i)=\nu_N(x_i)$ for each i and $a=b,\ c=d$ i.e., $M=N$. Thus, $I_R^S(M,N)$ is a valid discriminant measure of PFS M from PFS N.

Theorem 5.2 $J_R^S(M,N) = I_R^S(M,N)+I_R^S(N,M)$ *is the valid symmetric discriminant measure.*

Proof : The proof can be carried on the similar lines as the proof of Theorem 5.1.

5.1.1 Properties of Proposed Discriminant Measure

Theorem 5.3 *Consider $M,\ N,\ C$ be three Pythagorean fuzzy sets defined over universe of discourse X.*

(i) $I_R^S(M \cup N,\ M) + I_R^S(M \cap N,\ M) = I_R^S(N,\ M)$.

(ii) $I_R^S(M \cup N,\ C) + I_R^S(M \cap N,\ C) = I_R^S(M,\ C) + I_R^S(N, C)$.

(iii) $I_R^S(\overline{M \cup N},\ \overline{M \cap N}) = I_R^S(\overline{M} \cap \overline{N}, \overline{M} \cup \overline{N})$.

(iv) $I_R^S(M,\ \overline{M}) = I_R^S(\overline{M},\ M)$.

(v) $I_R^S(\overline{M},\ \overline{N}) = I_R^S(M,\ N)$.

(vi) $I_R^S(M,\ \overline{N}) = I_R^S(\overline{M},\ N)$.

(vii) $I_R^S(M,\ N) + I_R^S(\overline{M},\ N) = I_R^S(\overline{M},\ \overline{N}) + I_R^S(M,\ \overline{N})$.

Proof : First, we partition X into two sub-divisions X_1 and X_2 such that

$$X_1 = \{x_i \in X \mid M \subseteq N\},\ \text{i.e., } \mu_M(x_i) \le \mu_N(x_i), \nu_M(x_i) \ge \nu_N(x_i) \forall\ x_i \in X_1;$$

$$X_2 = \{x_i \in X \mid N \subseteq M\},\ \text{i.e., } \mu_M(x_i) \ge \mu_N(x_i),\ \nu_M(x_i) \le \nu_N(x_i) \forall\ x_i \in X_2.$$

Now,

(i) We have to prove $I_R^S(M \cup N, M) + I_R^S(M \cap N, M) = I_R^S(N, M)$. We consider

$$I_R^S(M \cup N, M) + I_R^S(M \cap N, M)$$
$$= \frac{R \times S}{n\ (S-R)} \sum_{i=1}^{n} \begin{bmatrix} \left(\mu_{M\cup N}(x_i)^{2S}\mu_M(x_i)^{2(1-S)} + \nu_{M\cup N}(x_i)^{2S}\nu_M(x_i)^{2(1-S)} + \pi_{M\cup N}(x_i)^{2S}\pi_M(x_i)^{2(1-S)}\right)^{\frac{1}{S}} \\ -\left(\mu_{M\cup N}(x_i)^{2R}\mu_M(x_i)^{2(1-R)} + \nu_{M\cup N}(x_i)^{2R}\nu_M(x_i)^{2(1-R)} + \pi_{M\cup N}(x_i)^{2R}\pi_M(x_i)^{2(1-R)}\right)^{\frac{1}{R}} \end{bmatrix}$$
$$+ \frac{R \times S}{n(S-R)} \sum_{i=1}^{n} \begin{bmatrix} \left(\mu_{M\cap N}(x_i)^{2S}\mu_M(x_i)^{2(1-S)} + \nu_{M\cap N}(x_i)^{2S}\nu_M(x_i)^{2(1-S)} + \pi_{M\cap N}(x_i)^{2S}\pi_M(x_i)^{2(1-S)}\right)^{\frac{1}{S}} \\ -\left(\mu_{M\cap N}(x_i)^{2R}\mu_M(x_i)^{2(1-R)} + \nu_{M\cap N}(x_i)^{2R}\nu_M(x_i)^{2(1-R)} + \pi_{M\cap N}(x_i)^{2R}\pi_M(x_i)^{2(1-R)}\right)^{\frac{1}{R}} \end{bmatrix} \tag{5.1.18}$$

$$= \frac{R \times S}{n(S-R)} \sum_{X_1} \begin{bmatrix} \left(\mu_{M\cup N}(x_i)^{2S}\mu_M(x_i)^{2(1-S)} + \nu_{M\cup N}(x_i)^{2S}\nu_M(x_i)^{2(1-S)} + \pi_{M\cup N}(x_i)^{2S}\pi_M(x_i)^{2(1-S)}\right)^{\frac{1}{S}} \\ -\left(\mu_{M\cup N}(x_i)^{2R}\mu_M(x_i)^{2(1-R)} + \nu_{M\cup N}(x_i)^{2R}\nu_M(x_i)^{2(1-R)} + \pi_{M\cup N}(x_i)^{2R}\pi_M(x_i)^{2(1-R)}\right)^{\frac{1}{R}} \\ +\left(\mu_{M\cap N}(x_i)^{2S}\mu_M(x_i)^{2(1-S)} + \nu_{M\cap N}(x_i)^{2S}\nu_M(x_i)^{2(1-S)} + \pi_{M\cap N}(x_i)^{2S}\pi_M(x_i)^{2(1-S)}\right)^{\frac{1}{S}} \\ -\left(\mu_{M\cap N}(x_i)^{2R}\mu_M(x_i)^{2(1-R)} + \nu_{M\cap N}(x_i)^{2R}\nu_M(x_i)^{2(1-R)} + \pi_{M\cap N}(x_i)^{2R}\pi_M(x_i)^{2(1-R)}\right)^{\frac{1}{R}} \end{bmatrix}$$
$$+ \frac{R \times S}{n(S-R)} \sum_{X_2} \begin{bmatrix} \left(\mu_{M\cup N}(x_i)^{2S}\mu_M(x_i)^{2(1-S)} + \nu_{M\cup N}(x_i)^{2S}\nu_M(x_i)^{2(1-S)} + \pi_{M\cup N}(x_i)^{2S}\pi_M(x_i)^{2(1-S)}\right)^{\frac{1}{S}} \\ -\left(\mu_{M\cup N}(x_i)^{2R}\mu_M(x_i)^{2(1-R)} + \nu_{M\cup N}(x_i)^{2R}\nu_M(x_i)^{2(1-R)} + \pi_{M\cup N}(x_i)^{2R}\pi_M(x_i)^{2(1-R)}\right)^{\frac{1}{R}} \\ +\left(\mu_{M\cap N}(x_i)^{2S}\mu_M(x_i)^{2(1-S)} + \nu_{M\cap N}(x_i)^{2S}\nu_M(x_i)^{2(1-S)} + \pi_{M\cap N}(x_i)^{2S}\pi_M(x_i)^{2(1-S)}\right)^{\frac{1}{S}} \\ -\left(\mu_{M\cap N}(x_i)^{2R}\mu_M(x_i)^{2(1-R)} + \nu_{M\cap N}(x_i)^{2R}\nu_M(x_i)^{2(1-R)} + \pi_{M\cap N}(x_i)^{2R}\pi_M(x_i)^{2(1-R)}\right)^{\frac{1}{R}} \end{bmatrix} \tag{5.1.19}$$

$$= \frac{R \times S}{n(S-R)} \sum_{X_1} \begin{bmatrix} \left(\mu_N(x_i)^{2S}\mu_M(x_i)^{2(1-S)} + \nu_N(x_i)^{2S}\nu_M(x_i)^{2(1-S)} + \pi_N(x_i)^{2S}\pi_M(x_i)^{2(1-S)}\right)^{\frac{1}{S}} \\ -\left(\mu_N(x_i)^{2R}\mu_M(x_i)^{2(1-R)} + \nu_N(x_i)^{2R}\nu_M(x_i)^{2(1-R)} + \pi_N(x_i)^{2R}\pi_M(x_i)^{2(1-R)}\right)^{\frac{1}{R}} \end{bmatrix}$$
$$+ \frac{R \times S}{n(S-R)} \sum_{X_2} \begin{bmatrix} \left(\mu_N(x_i)^{2S}\mu_M(x_i)^{2(1-S)} + \nu_N(x_i)^{2S}\nu_M(x_i)^{2(1-S)} + \pi_N(x_i)^{2S}\pi_M(x_i)^{2(1-S)}\right)^{\frac{1}{S}} \\ -\left(\mu_N(x_i)^{2R}\mu_M(x_i)^{2(1-R)} + \nu_N(x_i)^{2R}\nu_M(x_i)^{2(1-R)} + \pi_N(x_i)^{2R}\pi_M(x_i)^{2(1-R)}\right)^{\frac{1}{R}} \end{bmatrix} \tag{5.1.20}$$

$$= \frac{R \times S}{n(S-R)} \sum_{i=1}^{n} \begin{bmatrix} \left(\mu_N(x_i)^{2S}\mu_M(x_i)^{2(1-S)} + \nu_N(x_i)^{2S}\nu_M(x_i)^{2(1-S)} + \pi_N(x_i)^{2S}\pi_M(x_i)^{2(1-S)}\right)^{\frac{1}{S}} \\ -\left(\mu_N(x_i)^{2R}\mu_M(x_i)^{2(1-R)} + \nu_N(x_i)^{2R}\nu_M(x_i)^{2(1-R)} + \pi_N(x_i)^{2R}\pi_M(x_i)^{2(1-R)}\right)^{\frac{1}{R}} \end{bmatrix}$$
$$= I_R^S(N, M).$$

(ii) We consider

$$
\begin{aligned}
&I_R^S(M\cup\ N,C)+I_R^S(M\cap\ N,C)\\
&=\frac{R\times\ S}{n\ (S-R)}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\mu_{M\cup N}(x_i)^{2S}\mu_C(x_i)^{2(1-S)}+\nu_{M\cup N}(x_i)^{2S}\nu_C(x_i)^{2(1-S)}+\pi_{M\cup N}(x_i)^{2S}\pi_C(x_i)^{2(1-S)}\right)^{\frac{1}{S}}\\ -\left(\mu_{M\cup N}(x_i)^{2R}\mu_C(x_i)^{2(1-R)}+\nu_{M\cup N}(x_i)^{2R}\nu_C(x_i)^{2(1-R)}+\pi_{M\cup N}(x_i)^{2R}\pi_C(x_i)^{2(1-R)}\right)^{\frac{1}{R}}\end{array}\right]\\
&+\frac{R\times\ S}{n\ (S-R)}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\mu_{M\cap N}(x_i)^{2S}\mu_C(x_i)^{2(1-S)}+\nu_{M\cap N}(x_i)^{2S}\nu_C(x_i)^{2(1-S)}+\pi_{M\cap N}(x_i)^{2S}\pi_C(x_i)^{2(1-S)}\right)^{\frac{1}{S}}\\ -\left(\mu_{M\cap N}(x_i)^{2R}\mu_C(x_i)^{2(1-R)}+\nu_{M\cap N}(x_i)^{2R}\nu_C(x_i)^{2(1-R)}+\pi_{M\cap N}(x_i)^{2R}\pi_C(x_i)^{2(1-R)}\right)^{\frac{1}{R}}\end{array}\right]
\end{aligned}
\tag{5.1.21}
$$

$$
\begin{aligned}
&=\frac{R\times S}{n(S-R)}\sum_{X_1}\left[\begin{array}{c}\left(\mu_{M\cup N}(x_i)^{2S}\mu_C(x_i)^{2(1-S)}+\nu_{M\cup N}(x_i)^{2S}\nu_C(x_i)^{2(1-S)}+\pi_{M\cup N}(x_i)^{2S}\pi_C(x_i)^{2(1-S)}\right)^{\frac{1}{S}}\\ -\left(\mu_{M\cup N}(x_i)^{2R}\mu_C(x_i)^{2(1-R)}+\nu_{M\cup N}(x_i)^{2R}\nu_C(x_i)^{2(1-R)}+\pi_{M\cup N}(x_i)^{2R}\pi_C(x_i)^{2(1-R)}\right)^{\frac{1}{R}}\\ +\left(\mu_{M\cap N}(x_i)^{2S}\mu_C(x_i)^{2(1-S)}+\nu_{M\cap N}(x_i)^{2S}\nu_C(x_i)^{2(1-S)}+\pi_{M\cap N}(x_i)^{2S}\pi_C(x_i)^{2(1-S)}\right)^{\frac{1}{S}}\\ -\left(\mu_{M\cap N}(x_i)^{2R}\mu_C(x_i)^{2(1-R)}+\nu_{M\cap N}(x_i)^{2R}\nu_C(x_i)^{2(1-R)}+\pi_{M\cap N}(x_i)^{2R}\pi_C(x_i)^{2(1-R)}\right)^{\frac{1}{R}}\end{array}\right]\\
&+\frac{R\times S}{n(S-R)}\sum_{X_2}\left[\begin{array}{c}\left(\mu_{M\cup N}(x_i)^{2S}\mu_C(x_i)^{2(1-S)}+\nu_{M\cup N}(x_i)^{2S}\nu_C(x_i)^{2(1-S)}+\pi_{M\cup N}(x_i)^{2S}\pi_C(x_i)^{2(1-S)}\right)^{\frac{1}{S}}\\ -\left(\mu_{M\cup N}(x_i)^{2R}\mu_C(x_i)^{2(1-R)}+\nu_{M\cup N}(x_i)^{2R}\nu_C(x_i)^{2(1-R)}+\pi_{M\cup N}(x_i)^{2R}\pi_C(x_i)^{2(1-R)}\right)^{\frac{1}{R}}\\ +\left(\mu_{M\cap N}(x_i)^{2S}\mu_C(x_i)^{2(1-S)}+\nu_{M\cap N}(x_i)^{2S}\nu_C(x_i)^{2(1-S)}+\pi_{M\cap N}(x_i)^{2S}\pi_C(x_i)^{2(1-S)}\right)^{\frac{1}{S}}\\ -\left(\mu_{M\cap N}(x_i)^{2R}\mu_C(x_i)^{2(1-R)}+\nu_{M\cap N}(x_i)^{2R}\nu_C(x_i)^{2(1-R)}+\pi_{M\cap N}(x_i)^{2R}\pi_C(x_i)^{2(1-R)}\right)^{\frac{1}{R}}\end{array}\right]
\end{aligned}
\tag{5.1.22}
$$

$$
\begin{aligned}
&=\frac{R\times S}{n(S-R)}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\mu_M(x_i)^{2S}\mu_C(x_i)^{2(1-S)}+\nu_M(x_i)^{2S}\nu_C(x_i)^{2(1-S)}+\pi_M(x_i)^{2S}\pi_C(x_i)^{2(1-S)}\right)^{\frac{1}{S}}\\ -\left(\mu_M(x_i)^{2R}\mu_C(x_i)^{2(1-R)}+\nu_M(x_i)^{2R}\nu_C(x_i)^{2(1-R)}+\pi_M(x_i)^{2R}\pi_C(x_i)^{2(1-R)}\right)^{\frac{1}{R}}\end{array}\right]\\
&+\frac{R\times\ S}{n\ (S-R)}\sum_{i=1}^{n}\left[\begin{array}{c}\left(\mu_N(x_i)^{2S}\mu_C(x_i)^{2(1-S)}+\nu_N(x_i)^{2S}\nu_C(x_i)^{2(1-S)}+\pi_N(x_i)^{2S}\pi_C(x_i)^{2(1-S)}\right)^{\frac{1}{S}}\\ -\left(\mu_N(x_i)^{2R}\mu_C(x_i)^{2(1-R)}+\nu_N(x_i)^{2R}\nu_C(x_i)^{2(1-R)}+\pi_N(x_i)^{2R}\pi_C(x_i)^{2(1-R)}\right)^{\frac{1}{R}}\end{array}\right]\\
&=I_R^S(M,C)+I_R^S(N,C).
\end{aligned}
$$

Similarly, one can easily prove $(iii)-(vii)$.

5.2 Monotonic Nature of Proposed Discriminant Measure

The study of monotonic behaviour of the proposed discriminant measure has been carried out in an empirical way. Here, we take four different pairs of Pythagorean fuzzy sets $A=(P_1,P_2)$, $B=(P_3,P_4)$, $C=(P_5,P_6)$ and $D=(P_7,P_8)$ over the universe of discourse $X=\{x_1\ ,x_2\ ,x_3\ \}$:

$$P_1=\{(x_1,0.8,0.4),(x_2,0.7,0.6),(x_3,0.5,0.7)\};$$

$$P_2=\{(x_1,0.7,0.4),(x_2,0.6,0.5),(x_3,0.6,0.4)\}.$$

$$P_3 = \{(x_1, 0.2, 0.5), (x_2, 0.5, 0.7), (x_3, 0.3, 0.8)\};$$
$$P_4 = \{(x_1, 0.4, 0.7), (x_2, 0.5, 0.4), (x_3, 0.9, 0.3)\}.$$

$$P_5 = \{(x_1, 0.5, 0.6), (x_2, 0.2, 0.9), (x_3, 0.9, 0.4)\};$$
$$P_6 = \{(x_1, 0.4, 0.8), (x_2, 0.9, 0.3), (x_3, 0.7, 0.6)\}.$$

$$P_7 = \{(x_1, 0.4, 0.8), (x_2, 0.9, 0.4), (x_3, 0.5, 0.5)\};$$
$$P_8 = \{(x_1, 0.7, 0.6), (x_2, 0.5, 0.6), (x_3, 0.3, 0.8)\}.$$

Different values of parameters have been taken for detailed study and tabulated the computed values in Table 5.1. On the basis of the tabulated data and the plots are given below in Figure 5.1, it is quite clear that the proposed discriminant measure is a monotonically increasing function of R and S.

5.3 Computational Applications of Proposed Measure

In order to show the applicability of the proposed discriminant measure, we have considered three different fields of computational problems- pattern recognition, medical diagnosis, and decision-making.

5.3.1 Problem of Pattern Recognition

In this section, we have considered a well posed example taken from the existing literature ([70], [51]) to exhibit the applicability of the proposed discriminant measure.

Assume 3 existing patterns A_1, A_2 and A_3 representing the classes C_1, C_2 and C_3 respectively and being described by the following PFSs in $X = \{x_1, x_2, x_3\}$:

$$A_1 = \{(x_1, 0.3, 0.3), (x_2, 0.4, 0.4), (x_3, 0.4, 0.4)\};$$
$$A_2 = \{(x_1, 0.5, 0.5), (x_2, 0.1, 0.1), (x_3, 0.5, 0.5)\};$$
$$A_3 = \{(x_1, 0.5, 0.4), (x_2, 0.4, 0.5), (x_3, 0.3, 0.3)\}.$$

Also, suppose we have an unknown pattern Q

$$Q = \{(x_1, 0.4, 0.4), (x_2, 0.5, 0.5), (x_3, 0.2, 0.2)\};$$

Table 5.1: Values of R, S-norm Discriminant Measure

R	***S*= 0.15**				***S* = .25**				***S* = 0.4**			
	(P_1, P_2)	(P_3, P_4)	(P_5, P_6)	(P_7, P_8)	(P_1, P_2)	(P_3, P_4)	(P_5, P_6)	(P_7, P_8)	(P_1, P_2)	(P_3, P_4)	(P_5, P_6)	(P_7, P_8)
1.2	0.0275	0.0909	0.0933	0.0744	0.0958	0.1585	0.1642	0.1262	0.0757	0.2705	0.2845	0.2062
1.5	0.0354	0.1217	0.1271	0.0928	0.1227	0.2105	0.2216	0.1567	0.0971	0.3548	0.3779	0.2544
2	0.0484	0.1685	0.1781	0.1190	0.1667	0.2887	0.3068	0.1998	0.1323	0.4796	0.5140	0.3217
3.5	0.0800	0.2689	0.2839	0.1692	0.2694	0.4542	0.4811	0.2815	0.2151	0.7388	0.7864	0.4469
5	0.0991	0.3281	0.3441	0.1963	0.3289	0.5509	0.5792	0.3250	0.2637	0.8879	0.9370	0.5126
7	0.1138	0.3751	0.3912	0.2178	0.3736	0.6270	0.6554	0.3595	0.3004	1.0042	1.0531	0.5645
10	0.1259	0.4142	0.4308	0.2368	0.4095	0.6902	0.7191	0.3899	0.3302	1.1002	1.1495	0.6103
25	0.1446	0.4751	0.4933	0.2684	0.4641	0.7878	0.8194	0.4405	0.3756	1.2473	1.3005	0.6863
40	0.1498	0.4914	0.5104	0.2770	0.4791	0.8140	0.8467	0.4542	0.3881	1.2866	1.3414	0.7068
60	0.1529	0.5008	0.5201	0.2819	0.4880	0.8288	0.8622	0.4619	0.3956	1.3088	1.3647	0.7185
75	0.1541	0.5045	0.5241	0.2838	0.4917	0.8349	0.8685	0.4651	0.3987	1.3178	1.3741	0.7231
100	0.1554	0.5083	0.5280	0.2858	0.4955	0.8409	0.8748	0.4682	0.4018	1.3269	1.3835	0.7279
150	0.1568	0.5121	0.5320	0.2878	0.4994	0.8470	0.8812	0.4714	0.4050	1.3360	1.3930	0.7326
200	0.1574	0.5141	0.5340	0.2888	0.5013	0.8501	0.8844	0.4730	0.4067	1.3406	1.3978	0.7350
250	0.1578	0.5152	0.5352	0.2894	0.5025	0.8519	0.8863	0.4740	0.4076	1.3433	1.4007	0.7364
295	0.1581	0.5159	0.5360	0.2898	0.5032	0.8530	0.8875	0.4745	0.4082	1.3450	1.4024	0.7373
R	***S*= 0.50**				***S* = 0.70**				***S* = 0.95**			
	(P_1, P_2)	(P_3, P_4)	(P_5, P_6)	(P_7, P_8)	(P_1, P_2)	(P_3, P_4)	(P_5, P_6)	(P_7, P_8)	(P_1, P_2)	(P_3, P_4)	(P_5, P_6)	(P_7, P_8)
1.2	0.0958	0.3515	0.3736	0.2603	0.1372	0.5249	0.5674	0.3679	0.1908	0.7490	0.8185	0.4966
1.5	0.1227	0.4574	0.4909	0.3198	0.1753	0.6722	0.7302	0.4486	0.2428	0.9430	1.0315	0.6003
2	0.1667	0.6124	0.6598	0.4023	0.2369	0.8840	0.9601	0.5588	0.3260	1.2172	1.3269	0.7397
3.5	0.2694	0.9305	0.9935	0.5540	0.3778	1.3103	1.4043	0.7569	0.5111	1.7594	1.8863	0.9844
5	0.3289	1.1120	1.1760	0.6328	0.4574	1.5497	1.6431	0.8582	0.6124	2.0592	2.1820	1.1075
7	0.3736	1.2527	1.3160	0.6950	0.5162	1.7335	1.8245	0.9380	0.6857	2.2869	2.4045	1.2046
10	0.4095	1.3683	1.4318	0.7499	0.5629	1.8833	1.9737	1.0087	0.7433	2.4709	2.5865	1.2912
25	0.4641	1.5447	1.6127	0.8410	0.6334	2.1101	2.2056	1.1261	0.8292	2.7471	2.8682	1.4354
40	0.4791	1.5917	1.6616	0.8656	0.6527	2.1701	2.2681	1.1576	0.8527	2.8198	2.9439	1.4740
60	0.4880	1.6183	1.6894	0.8794	0.6642	2.2041	2.3036	1.1754	0.8668	2.8609	2.9867	1.4957
75	0.4917	1.6290	1.7006	0.8850	0.6690	2.2178	2.3179	1.1826	0.8727	2.8774	3.0040	1.5044
100	0.4955	1.6399	1.7119	0.8907	0.6739	2.2316	2.3322	1.1898	0.8788	2.8941	3.0214	1.5131
150	0.4994	1.6507	1.7232	0.8963	0.6789	2.2454	2.3467	1.1970	0.8849	2.9108	3.0389	1.5219
200	0.5013	1.6562	1.7289	0.8991	0.6814	2.2524	2.3540	1.2006	0.8880	2.9192	3.0477	1.5263
250	0.5025	1.6595	1.7324	0.9008	0.6829	2.2565	2.3583	1.2028	0.8898	2.9242	3.0529	1.5290
295	0.5032	1.6615	1.7345	0.9019	0.6839	2.2591	2.3610	1.2041	0.8909	2.9273	3.0562	1.5306

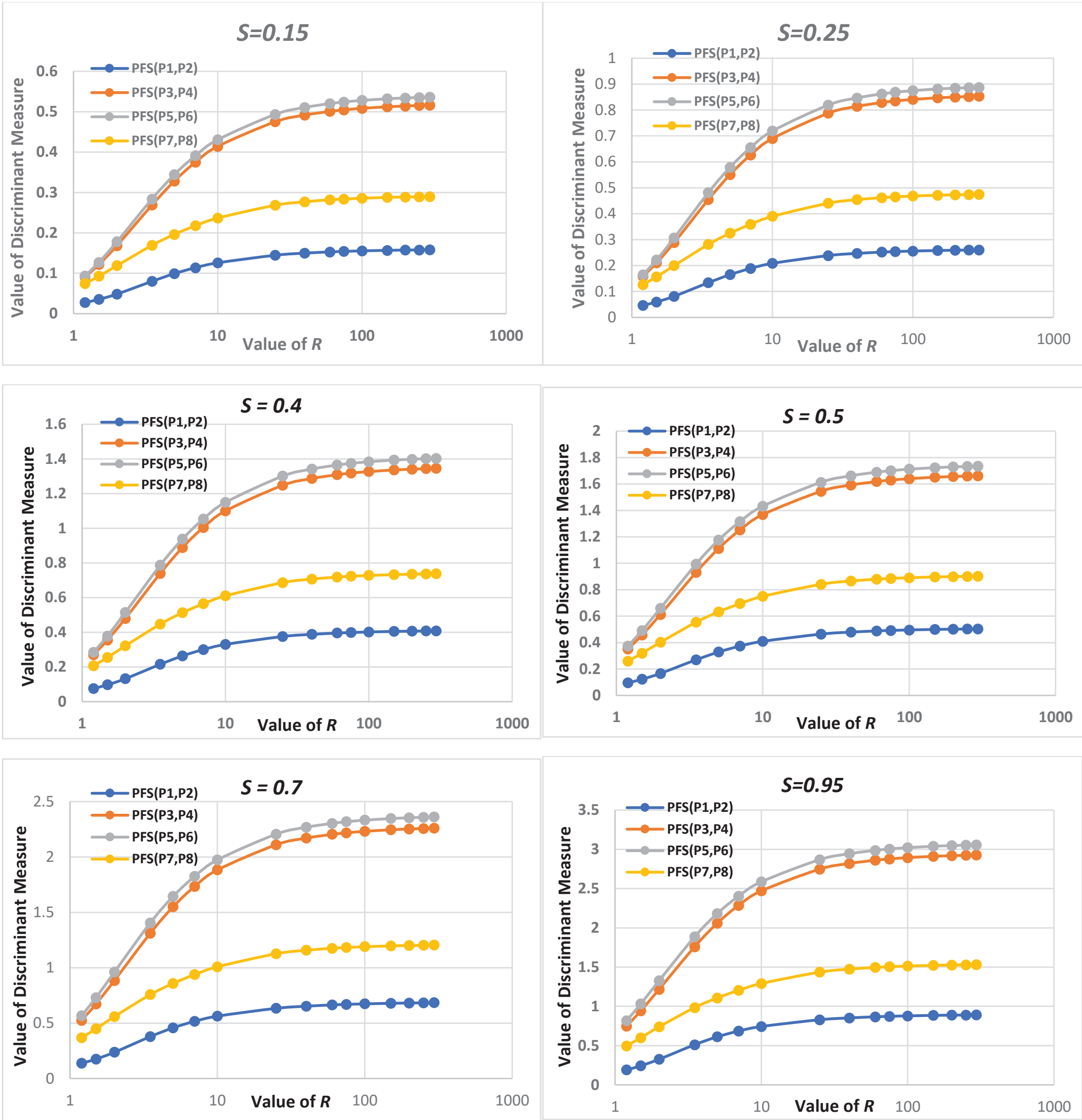

Figure 5.1: Monotonicity of the Proposed Discriminant Measure

which we have to allocate in one of the known class. For, this we have present the allocation procedure analogous to principle of minimum discriminant information[59] as:

$$\alpha^* = \min_{\alpha}(I_R^S(A_k, Q)). \tag{5.3.1}$$

In view of the values tabulated in Table 5.2, it may be noted that the unknown pattern Q has least discriminant value w.r.t the pattern A_3. Hence, the pattern Q must belong to the class C_3, which is perfectly consistent with the results achieved by [70] [51] [53] [49] [141].

Table 5.2: Values of $I_R^S(P_\alpha, Q)$, with $\alpha \in \{1, 2, 3\}$

	R	S	A_1	A_2	A_3
Q	0.1	10	0.1618	0.1475	**0.09325**
Q	0.9	10	1.1671	1.0211	**0.4706**

5.3.2 Medical Diagnosis Problem

Assume that a doctor needs to diagnose a patient P under a set of diagnoses $D = \{Viral\ fever,\ Malaria, Typhoid,\ Stomach\ problem, Chest\ problem\}$, with a set of symptoms $S = \{Temperature,\ Headache, Stomach\ pain,\ Cough,\ Chest\ pain\}$. The characteristic symptoms for the diagnoses and the symptoms for patient are provided in Table 5.3 and Table 5.4 respectively. Each component of the each table is being represented by the pair of numbers corresponding to the membership and non-membership values, respectively, e.g., in Table 5.3, $(\mu, \nu) = (0.4, 0.0)$ describes the temperature for viral fever. In order to have a proper diagnose, we evaluate the discriminant information measure $I_R^S(P, d_\alpha)$ between the patient's symptoms and the set of symptoms that are characteristic for each diagnose $d_\alpha \in D$, with $\alpha = \{1, 2, 3, 4, 5\}$. Similar to the equation (5.3.1), the proper diagnose d_α for the patient P may be based on the following analogous equation:

$$\alpha^* = arg \min_\alpha (I_R^S(P, d_\alpha)). \tag{5.3.2}$$

Table 5.3: Symptoms characteristic for the diagnoses considered

	Viral Fever	Malaria	Typhoid	Stomach Prob.	Chest Prob.
Temperature	$(0.4, 0.0)$	$(0.7, 0.0)$	$(0.3, 0.3)$	$(0.1, 0.7)$	$(0.1, 0.8)$
Headache	$(0.3, 0.5)$	$(0.2, 0.6)$	$(0.6, 0.1)$	$(0.2, 0.4)$	$(0.0, 0.8)$
Stomach Pain	$(0.1, 0.7)$	$(0.0, 0.9)$	$(0.2, 0.7)$	$(0.8, 0.0)$	$(0.2, 0.8)$
Cough	$(0.4, 0.3)$	$(0.7, 0.0)$	$(0.2, 0.6)$	$(0.2, 0.7)$	$(0.2, 0.8)$
Chest Pain	$(0.1, 0.7)$	$(0.1, 0.8)$	$(0.1, 0.9)$	$(0.2, 0.7)$	$(0.8, 0.1)$

Table 5.4: Symptoms for the diagnose under consideration

	Temperature	Headache	Stomach Pain	Cough	Chest Pain
P	$(0.8, 0.1)$	$(0.6, 0.1)$	$(0.2, 0.8)$	$(0.6, 0.1)$	$(0.1, 0.6)$

Therefore, the patient is diagnosed with symptoms which have the least value of the discriminant measure from patient's symptoms. The results for the considered patient P have been computed and presented in the Table 5.5.

Table 5.5: Values of $I_R^S(P, d_\alpha)$

	Viral Fever	Malaria	Typhoid	Stomach Prob.	Chest Prob.
P	**0.1641**	0.1733	0.2782	1.7484	1.7837

Hence, the patient P is suffering from the viral fever. It may be observed that the results obtained through the proposed method are perfectly consistent with the results achieved by Wei & Wei [49] and Garg [52].

Comparative Remarks: It may be observed that the proposed method is found to be perfectly competent to provide the desired result with an added advantage of the parameters involvement in the proposed discriminant measure. The parameters may provide a better variability in the selection of a discriminant measure for achieving a better specificity and accuracy.

5.3.3 Multi-criteria Decision Making Problem

The main objective of the multi-criteria decision making problem is to select the optimal/best alternative out of the m feasible available alternatives, i.e., $Z = \{Z_1, Z_2, \ldots, Z_m\}$ based on certain laid down criteria n criteria $O = \{o_1, o_2, \ldots, o_n\}$. For this, first we take the appraisal values of an alternative $z_i\,(i = 1, 2, 3, \ldots, m)$ w.r.t the criteria $o_j\,(j = 1, 2, 3, \ldots, n)$ is given by $z_{ij} = (p_{ij}, q_{ij})$, satisfying $0 \le p_{ij} \le 1$, $0 \le q_{ij} \le 1$ and $0 \le p_{ij} + q_{ij} \le 1$ with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Procedural Steps of Algorithm for MCDM Problem:

Step 1: Thus, the above problem can be modeled by representing it through the following Pythagorean fuzzy decision matrix:

$$R = (p_{ij}, q_{ij})_{m\times n} = (z_{ij}) = \begin{array}{ccccc} & o_1 & o_2 & \cdots & o_n \\ z_1 & (p_{11}, q_{11}) & (p_{12}, q_{12}) & \cdots & (p_{1n}, q_{1n}) \\ z_2 & (p_{21}, q_{21}) & (p_{22}, q_{22}) & \cdots & (p_{2n}, q_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_m & (p_{m1}, q_{m1}) & (p_{m2}, q_{m2}) & \cdots & (p_{mn}, q_{mn}) \end{array}$$

Step 2: In order to maintain homogeneity in the criterions, we need to transform the decision matrix obtained in Step 1. Thus the decision matrix $A = [a_{ij}]_{m\times n}$ is converted into a new decision matrix, say, $B = [b_{ij}]_{m\times n}$ where b_{ij} is given by

$$b_{ij} = (\mu_{ij}, \nu_{ij}) = \begin{cases} a_{ij} & \text{for benefits criteria ;} \\ a_{ij}^c & \text{for cost criteria ;} \end{cases} \tag{5.3.3}$$

where $B = [b_{ij}]_{m\times n}$ representing the alternatives in the form of

$$Z_i = \{(o_j, \mu_{ij}, \nu_{ij}) | o_j \in O\}; \quad i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n. \tag{5.3.4}$$

Step 3: Compute the best preferred solution as

$$Z^+ = \{\sup(\mu_{ij}(Z_i)), \ \inf(\nu_{ij}(Z_i))\} \ \ i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n. \tag{5.3.5}$$

Step 4: Evaluate the value of the discriminant measure of alternatives $Z_i's$ from Z^+ using equation (5.1.2).

Step 5. Based on the values obtained in Step 4, we can determine the optimal ranking order of the alternatives. The alternative with the least value of discriminant measure is supposed to be the best alternative.

Example 5.1 *Assume a real estate company needs to procure the material for its upcoming project. The company advertises for receiving the tenders for purchasing the required material. Let us suppose that there are 5 suppliers in the market, say, Z_1, Z_2, Z_3, Z_4 and Z_5 and six criterions for supplier selection which company has fixed, say, o_1 (quality of material), o_2 (price), o_3 (services), o_4 (delivery), o_5 (technical support) and o_6 (behavior).*
Then for the above MCDM problem, the Pythagorean fuzzy decision matrix $A = [a_{ij}]_{m\times n}$ may be given by the following Table 5.6.

Table 5.6: Pythagorean Fuzzy Decision Matrix

	o_1	o_2	o_3	o_4	o_5	o_6
Z_1	$(0.4, 0.5)$	$(0.8, 0.1)$	$(0.7, 0.3)$	$(0.6, 0.2)$	$(0.5, 0.4)$	$(0.3, 0.4)$
Z_2	$(0.7, 0.2)$	$(0.5, 0.3)$	$(0.3, 0.4)$	$(0.8, 0.1)$	$(0.2, 0.4)$	$(0.4, 0.5)$
Z_3	$(0.6, 0.1)$	$(0.7, 0.3)$	$(0.6, 0.2)$	$(0.4, 0.1)$	$(0.3, 0.4)$	$(0.8, 0.2)$
Z_4	$(0.5, 0.4)$	$(0.3, 0.4)$	$(0.8, 0.1)$	$(0.7, 0.2)$	$(0.6, 0.1)$	$(0.7, 0.1)$
Z_5	$(0.4, 0.3)$	$(0.7, 0.1)$	$(0.5, 0.2)$	$(0.9, 0.1)$	$(0.8, 0.1)$	$(0.6, 0.4)$

The computational steps for the above stated Example 5.1 are as follows:

Table 5.7: Transformed Pythagorean Fuzzy Decision Matrix

	o_1	o_2	o_3	o_4	o_5	o_6
Z_1	$(0.4, 0.5)$	$(0.1, 0.8)$	$(0.7, 0.3)$	$(0.6, 0.2)$	$(0.5, 0.4)$	$(0.3, 0.4)$
Z_2	$(0.7, 0.2)$	$(0.3, 0.5)$	$(0.3, 0.4)$	$(0.8, 0.1)$	$(0.2, 0.4)$	$(0.4, 0.5)$
Z_3	$(0.6, 0.1)$	$(0.3, 0.7)$	$(0.6, 0.2)$	$(0.4, 0.1)$	$(0.3, 0.4)$	$(0.8, 0.2)$
Z_4	$(0.5, 0.4)$	$(0.4, 0.3)$	$(0.8, 0.1)$	$(0.7, 0.2)$	$(0.6, 0.1)$	$(0.7, 0.1)$
Z_5	$(0.4, 0.3)$	$(0.1, 0.7)$	$(0.5, 0.2)$	$(0.9, 0.1)$	$(0.8, 0.1)$	$(0.6, 0.4)$

1. We find the transformed matrix by using equation (5.3.3) and the transformed Pythagorean fuzzy decision matrix is given in the following Table 5.7.

2. In this step, we dtermine the best preferred solution by using equation (5.3.5) as follows:

$$Z^+ = \{(0.7,\ 0.1), (0.4, 0.3), (0.8,\ 0.1), (0.9, 0.1), (0.8,\ 0.1), (0.8,\ 0.1)\}. \tag{5.3.6}$$

3. We compute the discriminant measures between $Z_i's$ $(i = 1, \ldots, 5)$ and Z^+ using equation (5.1.2) and the values are tabulated in the following Table 5.8.

Table 5.8: Evaluated values of Discriminant Measure between $Z_i's$ and Z^+

$I_R^S(Z_1, Z^+)$	0.7791
$I_R^S(Z_2, Z^+)$	0.6438
$I_R^S(Z_3, Z^+)$	0.3395
$I_R^S(Z_4, Z^+)$	0.2042
$I_R^S(Z_5, Z^+)$	0.3319

4. In view of the values obtained in above step, the ranking of the alternatives is as:

$$Z_4 > Z_5 > Z_3 > Z_2 > Z_1;$$

and Z_4 is the optimal/best available alternative.

5.4 Comparative Analysis

We compare the performance of the proposed method for decision making with the existing TOPSIS [21] [139] and the MOORA method [130].

5.4.1 Comparison of Proposed Method with TOPSIS Technique

Incorporating the proposed discriminant measure in the TOPSIS technique, the procedural steps may be given as follows:

Step 1. Construct the matrix $A = [a_{ij}]_{m\times n}$ called Pythagorean fuzzy decision matrix, where $a_{ij} = (\mu_{ij}, \nu_{ij})$ representing the degree of membership and non-membership respectively.

Step 2. Normalize the Pythagorean fuzzy decision matrix constructed in Step 1 as follows:

$$\mu'_{ij} = \frac{\mu_{ij}}{\sqrt{\sum_{i=1}^{m}(\mu_{ij})^2}} \quad \text{and} \quad \nu'_{ij} = \frac{\nu_{ij}}{\sqrt{\sum_{i=1}^{m}(\nu_{ij})^2}}. \tag{5.4.1}$$

Let us take $B = [b_{ij}]_{m\times n}$, where $b_{ij} = (\mu'_{ij}, \nu'_{ij})$.

Step 3. Formulate the weighted normalized Pythagorean fuzzy decision matrix as: $W = [w_{ij}]_{m\times n}$, where $w_{ij} = u_i b_{ij}$; $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. In this MCDM problem under consideration, we have taken $u_i = 1\ \forall i = 1, 2, \ldots, m$. It may be noted that u_i's are components of the weight vector.

Step 4. Evaluate the best preferred solution, i.e., Z^+ and the worst solution, i.e., Z^- as:

$$\begin{aligned} Z^+ &= \{\alpha_1^+, \alpha_2^+, \ldots, \alpha_n^+, \}; \\ Z^- &= \{\alpha_1^-, \alpha_2^-, \ldots, \alpha_n^-, \}; \end{aligned} \tag{5.4.2}$$

where $\alpha_j^+ = (\sup \mu_{ij}(Z_i), \inf \nu_{ij}(Z_i))$ and $\alpha_j^- = (\inf \mu_{ij}(Z_i), \sup \mu_{ij}(Z_i))$.

Step 5. Determine the discriminant measures of Z_i's $\forall$ $(i = 1, 2, \ldots, m)$ from Z^+ and Z^- respectively by taking the proposed measure (5.1.2) into account.

Step 6. Compute the coefficient of relative closeness, i.e, C_i's , $(i = 1, 2, \ldots m)$ as:

$$C_i = \frac{I_R^S(Z_i, Z^-)}{I_R^S(Z_i, Z^+) + I_R^S(Z_i, Z^-)}. \tag{5.4.3}$$

Step 7. Rank the alternatives Z_i $(i = 1, 2, \ldots m)$ with respect to the coefficient of relative closeness.

The computational values using the above steps for the MCDM problem by TOPSIS technique are as:

1. First, we consider the transformed Pythagorean fuzzy decision matrix as given in Table 5.7.

2. Normalizing the above matrix using (5.4.1), we have the following Pythagorean fuzzy normalized decision matrix as given in Figure 5.2.

Figure 5.2: Pythagorean Fuzzy Normalized Decision Matrix

	o_1	o_2	o_3	o_4	o_5	o_6
Z_1	$(0.3357, 0.6742)$	$(0.1667, 0.5714)$	$(0.5175, 0.5145)$	$(0.3825, 0.6030)$	$(0.4256, 0.4815)$	$(0.2274, 0.5080)$
Z_2	$(0.5874, 0.2697)$	$(0.5000, 0.3571)$	$(0.2218, 0.6860)$	$(0.5101, 0.3015)$	$(0.1703, 0.7223)$	$(0.3032, 0.6350)$
Z_3	$(0.5035, 0.1348)$	$(0.5000, 0.5000)$	$(0.4435, 0.3430)$	$(0.43530, 0.3015)$	$(0.2554, 0.4815)$	$(0.6065, 0.2540)$
Z_4	$(0.4196, 0.5394)$	$(0.6667, 0.2143)$	$(0.5914, 0.1715)$	$(0.4463, 0.6030)$	$(0.5108, 0.1204)$	$(0.5307, 0.1270)$
Z_5	$(0.3357, 0.4045)$	$(0.1667, 0.5000)$	$(0.3696, 0.3430)$	$(0.5738, 0.3015)$	$(0.6810, 0.4781)$	$(0.4549, 0.5080)$

3. Evaluate the best preferred solution Z^+ and the worst solution Z^- using equation (5.4.2):

$$Z^+ = \{(0.5874, 0.1348), (0.6667, 0.2143), (0.5914, 0.1715),$$
$$(0.5738, 0.3015), (0.6810, 0.1204), (0.6065, 0.1270)\};$$
$$Z^- = \{(0.3357, 0.6742), (0.1667, 0.5714), (0.2218, 0.6860),$$
$$(0.2550, 0.6030), (0.1703, 0.7223), (0.2274, 0.6350)\}.$$

4. The computed values of the discriminant measure I_R^S of Z_i's from Z^+ and Z^- is given in Table 5.9.

Table 5.9: Computed values of $I_R^S(Z_i, Z^+)$ and $I_R^S(Z_i, Z^-)$

	$I_R^S(Z_i, Z^+)$	$I_R^S(Z_i, Z^-)$
Z_1	0.8181	0.1308
Z_2	0.8908	0.1686
Z_3	0.2931	0.2578
Z_4	0.2151	0.4545
Z_5	0.5068	0.3033

5. Determine the values of coefficients of relative closeness by using equation (5.4.3) as follows:

$$C_1 = 0.1378;\ C_2 = 0.1591;\ C_3 = 0.468;$$
$$C_4 = 0.6788;\ C_5 = 0.3744.$$

6. Finally, the ranking of the alternatives according to the values of the coefficients of relative closeness, i.e., $C_i^{'}s\ i = 1, 2, \ldots, 5$ can be performed. The sequence of alternatives so obtained is given by

$$Z_4 > Z_3 > Z_5 > Z_2 > Z_1.$$

Therefore, Z_4 is the best alternative among all $Z_i^{'}s$ $(i = 1, 2, \ldots, 5)$.

5.4.2 Comparison of Proposed Method with MOORA

Incorporating the proposed discriminant measure in the MOORA technique, the procedural steps may be given as follows:

Step 1. First three computational steps are same as of the TOPSIS technique.

Step 2. Compute the best preferred solution, i.e., Z^+ and the worst solution, i.e., Z^- from the Table 5.2 given by

$$Z^+ = \{\alpha_1^+, \alpha_2^+, \ldots, \alpha_n^+,\} = (\max_i(\mu_{ij}), \min_i(\nu_{ij}));$$
$$Z^- = \{\alpha_1^-, \alpha_2^-, \ldots, \alpha_n^-,\} = (\min_i(\mu_{ij}), \max_i(\nu_{ij})); \tag{5.4.4}$$

for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 3. Evaluate the value of overall performance $\forall Z_i$, $(i = 1, 2, \ldots m)$ by using equation(5.1.2) as $I_R^S(Z^+, Z^-)$.

Step 4. Finally, the ranking of the alternatives with respect to the computed values of the overall performance and the best alternative is the one which has the least value of the overall performance among all the alternatives.

The computational values using the above steps for the MCDM problem by MOORA technique are as:

1. Determine the values of Z^+ and Z^- from equations (5.4.4), which are provided in the Table 5.10.

Table 5.10: Computed values of Z^+ and Z^-

	Z^+	Z^-
Z_1	$(0.5175, 0.4815)$	$(0.1667, 0.6742$
Z_2	$(0.5874, 0.2697)$	$(0.1703, 0.7223)$
Z_3	$(0.6065, 0.1348)$	$(0.2550, 0.4815)$
Z_4	$(0.6667, 0.1204)$	$(0.4196, 0.6030)$
Z_5	$(0.6810, 0.3015)$	$(0.1667, 0.5080)$

2. Compute the values of the proposed discriminant measure $I_R^S(Z^+, Z^-)$, which are given in Table 5.11.

3. Finally, the ranking of the alternatives according to the computed values of $I_R^S(Z^+, Z^-)$. The sequence of alternatives so obtained is given by

$$Z_4 > Z_3 > Z_2 > Z_1 > Z_5.$$

Table 5.11: Evaluated values of $I_R^S(Z^+, Z^-)$

	$I_R^S(Z^+, Z^-)$	*Ranking*
Z_1	0.6013	4
Z_2	0.7963	3
Z_3	0.3738	2
Z_4	0.1683	1
Z_5	1.1173	5

Thus, Z_4 is the best alternative among all $Z_i^{'}s$ $(i = 1, 2, \ldots, 5)$.

Table 5.12: Ranking of the alternatives with Different Techniques

	Proposed Technique	TOPSIS	MOORA
Z_1	5	5	4
Z_2	4	4	3
Z_3	3	2	2
Z_4	1	1	1
Z_5	2	3	5

In view of above discussions, we find that the Z_4 is the best alternative in all the discussed techniques. It may also be observed that mutual fluctuation is present in the final ranking of the other alternatives while using different techniques (TOPSIS, MOORA) reflecting in Table 5.12. This is because of the fact that the different algorithms have their different perspectives and techniques as an ideal/universal ranking is totally a dependent concept on various influencing factors.

5.4.3 Observations and Advantages of the Proposed Method

On the basis of comparative analysis carried above, some important observations and remarks are being stated as follows:

- In TOPSIS technique, the alternatives used are to be arranged based on the coefficient of relative closeness whose values are lying between 0 and 1. However, in MOORA technique, the alternatives are ranked based on their overall performance.
- While in the method which is proposed in this chapter, each alternative is evaluated in reference with each laid down criterion individually before declaring the best/optimal alternative.

- It may be observed that the proposed method recommends the specific input as well as a specific procedure at the same time while this isn't the situation with different techniques. Also, the proposed method is straightforward to apply and amounts to have less calculations than other well known MCDM methods which have a little imprecise procedure to fetch a conclusion.

5.5 Conclusion

In this chapter, we have successfully proposed a new parametric (R, S)-norm discriminant measure for Pythagorean fuzzy set along with the proof of its validity and also studied its monotonic behavior w.r.t parameters R & S. The applicability of the proposed discriminant measure has also been worked out and illustrated through a numerical example in the computational application fields of pattern recognition, medical diagnosis. Also, an algorithm for multi-criteria decision-making problem has been well proposed and successfully implemented with the help of numerical example.

Chapter 6

Modified VIKOR and TOPSIS Method with Pythagorean Fuzzy Information Measures

6.1 Introduction

The fundamental natural resources such as renewable energy and fossil fuel plays a significant role in the socio-economic growth of a country. It has no exception that throughout the world, the demand of energy is increasing significantly with time. The stock of natural resources exhaust rapidly because of the major dependency on the fossil fuels which also leads to the emission of carbon dioxide and harm the environment. The introduction of electrification technique has significantly reduced the pollution component, but eventually could not be considered as a promising solution. Despite of this, the sustainable energy sources such as hydrogen energy, solar energy, wind energy, bio-fuel, geothermal, biomass energy, etc. can also be utilized in practical purposes.

In view of the present scenario, the existing technology, financial implications and eco-friendly prospects, the utilization of hydrogen energy is considered to be one of the best alternative source of energy. The major advantage of using hydrogen as a source of energy is mainly two fold - first, it is an extreme heat-burning gas; and second it does not release any toxic gas (e.g. CO_2, SO_2 and NO_2) on combustion. Since the hydrogen can be obtained from water (electrolysis) and solar energy (solar hydrogen), therefore, we can have an ample and endless source of hydrogen energy for the society and its need. Hence, the consideration of hydrogen energy is supposed to be a kind of clean renewable energy having perfectly zero emissions for the future prospects and it has received due attention of the researchers in recent past [87]. Various researchers dealt with issues of felicitating the hydrogen energy for the energy balance [91]. For the sake of electricity production, alternative energy sources - hydrogenated fuels have also been utilized [128]. Juste [48] experimented with hydrogen injection as an augmented fuel and investigated the gas turbine combustion chambers.

In the area of electromobility, the hydrogen energy has been potentially recognized as fuel cell. The fuel cell electrical vehicles (FCEVs)/battery vehicles uses a fuel cell (where the hydrogen is used) instead of a battery. Though the FCEV's cost is not practical and repressive in current time but the people are somewhat prepared/in-transit for taking the joint responsibility so that the ecological damage could be controlled [89] [111] [112] [85]. In addition to the cost limitation of FCEVs, there is another inter-correlated issue of hydrogen refuelling stations (HRSs). It is certainly easy to understand that the utilization of hydrogen energy in electro-mobility sector is a kind of wise investment for a significantly long time. Therefore, many decision makers [11] [88] [26] [45] have emphasized on the synchronized development of FCEVs and HRSs because of its advantageous features in all respects.

From few decades, various researchers and decision makers have put down their significant focus on the selection issues of renewable energy sources/technologies, specially on hydrogen energy, which has always been a major task. The task of choosing the right and most appropriate site for such sustainable energy comprises of demographic view point, socio-economic factor and infrastructure. The decision makers also need to focus on all the inter-related quantitative and qualitative factors. Therefore, the process of multi-criteria decision-making plays a critical role to model the structure of the available resources and criteria for such kinds of complex real life problem. A formal process of site selection can be well understood through Figure 6.1.

The decision-making algorithms certainly enhance the capabilities of the decision makers to moderate the content of decisions in terms of their rationality and efficiency in a better sense. The process of site selection for hydrogen power plant can be modeled as a multiple-criteria decision-making (MCDM) problem where various available inter-conflicting attributes can be explored. In general, the indicators affecting the available alternatives and their criteria/weights should

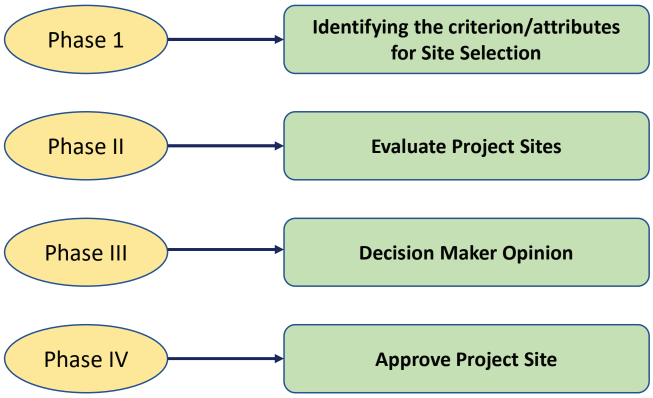

Figure 6.1: Phases of Site Selection Process

be quantitative for each available option. In human sense, there is always a constraint of inaccuracy and ambiguity which certainly limits to obtain an exact and precise value for evaluating the outcomes of the alternatives. In order to deal such incompleteness in the information, linguistic assessment by the experts in terms of fuzzy numbers (FNs) [82] have been found to be useful, effective and convenient approach for better handling.

In recent years, various researchers have extensively studied different information measures (similarity measures, entropy, distance measures, discriminant measures etc.) because of their wider applicability in the field of decision-making problems, pattern recognition, sales analysis, financial services, medical diagnosis etc. The theory of fuzzy sets/intuitionistic fuzzy [68][67] have been applied to model uncertainties and hesitancy inherent in many practical circumstances. Yager [106] proposed the Pythagorean fuzzy set (PFS) which is a useful generalization of IFS, characterized by degree of membership/non-membership fulfilling the inequality that the sum of squares of these values ≤ 1.

6.2 Literature Survey

A brief literature survey in connection with multi-criteria decision-making model for various types of renewable energy resources has been presented in this section. Wang et al.[20] presented a MCDM approach using fuzzy analytic network process (FANP) along with TOPSIS for the selection of nuclear power plant site in Vietnam. Recently, Sedady et al. [43] proposed the MCDM model for constructing renewable power plants by defining the actual priority of technology, socio-economic aspect, political and ecological aspects. A review paper [127] considering the above stated aspects in the estimated power-to-gas conversion along with an extended version to nuclear-assisted renewable hydrogen has also been reported.

For the installation of wind energy plants, Biswal and Shukla [47] proposed new methodologies for the selection of most suitable sites. Also, Pamucar [34] jointly utilized the concept of Geographical Information Systems (GIS), MCDM approach of Best-Worst Method (BWM) & multiple attribute decision making approach of ideal-real comparative analysis for the selection of wind turbine sites. Keeping the classical aspects of technology, ecology, economy and geographical point of view, Noorollahi [145] presented a MCDM support system for wind energy location selection with the help of GIS. Also, using GIS and fuzzy logic, Borah et al. [66] presented a framework for the site selection of wind turbines to achieve the optimum energy output.

A MCDM model for the selection of a solar plant site in Vietnam has been presented by Wang et al. [22] where "*fuzzy analytic hierarchy process (FAHP)*" and "*data envelopment analysis (DEA)*" have been utilized to find the best appropriate and suitable site considering both the qualitative and quantitative aspects. Wang et al.[23] developed the MCDM approach for solid wastes to energy plant sites in Vietnam using FANP and TOPSIS. Aktas et al. [2] developed a hybrid MCDM method using the notion of hesitant fuzzy sets for the selection of solar power plant site.

He et al. [17] proposed a hydrogen station optimization model for the setup of hydrogen-energy expressway in order to reduce the production cost. Lewandowska-Smierzchalska et al. [62] presented a decision-making model based on the popularly used AHP method to obtain the potential hydrogen storage sites in Poland. Deveci [73] proposed a MCDM approach for the selection of hydrogen storage sites based on the information provided in the interval type-2 hesitant fuzzy setup and carried out the sensitivity analysis to show the effectiveness of the proposed methodology. Narayanamoorthy et al. [117] proposed normal wiggly dual hesitant fuzzy set (NWDHFS) along with its score function and used in the MCDM method to find out the best hydrogen storage sites. Messaoudi [33] proposed an integrated framework with the combination of MCDM and GIS to evaluate the best location for the solar hydrogen production installation system. Karatas [76] provided a new methodology by integrating the FAHP and weighted fuzzy axiomatic design to select the hydrogen energy storage site in Turkey and carried out the sensitivity analysis in order to validate the robustness of proposed method. Tian et al. [81] utilized the AHP and TOPSIS method to explore the optimal region for developing the hydrogen energy applications and presented a case study for its functioning in terms of industrial and cultural prospects. Lin et al. [103] studied the different hydrogen station location models available in the literature and compared their strengths & weaknesses in a comprehensive manner.

Due to the growing complexity in the decision-making processes and variability in the human's perceptions, the notion of Pythagorean fuzzy sets have received significant attention of various researchers. Yager and Abbasov [105] established the connection of Pythagorean fuzzy numbers (PFNs) with the complex numbers and studied its utility in the decision-making process. Thereafter, Zhang and Xu [139] presented the modified TOPSIS method for solving the decision-making problems by incorporating the information in the form of PFN. Also, Yager [107] studied various aggregation operators and presented its utility for solving decision-making problem in Pythagorean fuzzy setup. Further, Ma and Xu [147] introduced some Pythagorean fuzzy symmetric operators and applied in solving decision-making problems. Considering the concept of similarity measures for Pythagorean fuzzy sets, Zhang [138] presented a novel approach to solve the MCDM problems. In order to understand the perception of the decision makers in solving the problems, Ren et al. [97] provided the Pythagorean fuzzy-Portuguese for interactive multi-criteria decision making approach. In order to demonstrate the eco-friendly energy methodologies with negative identical individual and infeasible criteria, the VIKOR method was given by Rani et al. [96] under Pythagorean fuzzy setup. Here the joint utility of every alternative is computed in terms of the developed discriminant measure for the Pythagorean fuzzy sets for selecting the renewable energy methodologies. Also, the applicability and dependability issues of the proposed approach have been duly discussed. Various other researchers have utilized the notion of Pythagorean fuzzy information in different capacities in the available literature.

From the above discussions, we note that all the intuitionistic fuzzy degrees are the special case of the Pythagorean fuzzy degrees, which indicates that the PFS proves to be more efficient to deal with vagueness, impreciseness and incompleteness present in the information than IFS. Certainly, the notion of PFS is in a more general frame work than IFS because the wider value of the degree of membership enables to have broader utility. In the present manuscript, Pythagorean fuzzy information measures ((R, S)-Norm entropy and discriminant measures) based MCDM techniques have been proposed and utilized for hydrogen energy plant site selection problem under a wider sense of fuzzy information. It may be noted that notions of parametric Pythagorean fuzzy entropy and Pythagorean fuzzy discriminant measures have not been utilized in the available literature and no study is available in connection with the hydrogen energy resources. In the present work, we have implemented the Pythagorean fuzzy information measures in bi-parametric form in the decision-making process format for renewable energy site selection problem with contrast.

The organizational structure of the present chapter is as follows: In Section 6.3, we have proposed a novel MCDM approach based on (R, S)-Norm Pythagorean fuzzy information measures implemented with VIKOR and TOPSIS methods. The problem of hydrogen power plant site selection has been appropriately dealt with the proposed methodologies in Section 6.4. In Section 6.5, we have provided the comparative analysis of the proposed methodologies with the existing literature in detail along with important remarks. Finally, Section 6.6 presents the concluding remarks of proposed research with some possible scope for future work.

6.3 Pythagorean Fuzzy Based MCDM Approach Utilizing (R, S)-Norm Information Measures

In this section, we propose two modified multi-criteria decision-making approaches based on VIKOR and TOPSIS method by incorporating the notion of (R, S)-Norm Pythagorean fuzzy entropy and respective (R, S)-Norm Pythagorean fuzzy discriminant measure. For the sake of clarity and better understanding, we present the basic structure of Pythagorean fuzzy information measures as follows:

Recently, Guleria and Bajaj [5] [7] proposed the following (R, S)-Norm Pythagorean fuzzy entropy for a Pythagorean fuzzy set $M \in PFS(U)$:

$$H_R^S(M) = \frac{R \times S}{(R-S)} \sum_{i=1}^{n} \frac{1}{n} \Big[\left(\mu_M(x_i)^{2S} + \nu_M(x_i)^{2S} + \pi_M(x_i)^{2S}\right)^{\frac{1}{S}} - \left(\mu_M(x_i)^{2R} + \nu_M(x_i)^{2R} + \pi_M(x_i)^{2R}\right)^{\frac{1}{R}} \Big]; \qquad (6.3.1)$$

and subsequently, for two Pythagorean fuzzy sets M and $N \in PFS(U)$, proposed (R, S)-Norm Pythagorean fuzzy discriminant measure as follows:

$$I_R^S(M, N) = \frac{R \times S}{n(S-R)} \sum_{i=1}^{n} \Big[\Big(\mu_M(x_i)^{2S} \mu_N(x_i)^{2(1-S)} + \nu_M(x_i)^{2S} \nu_N(x_i)^{2(1-S)} + \pi_M(x_i)^{2S} \pi_N(x_i)^{2(1-S)}\Big)^{\frac{1}{S}} - \Big(\mu_M(x_i)^{2R} \mu_N(x_i)^{2(1-R)} + \nu_M(x_i)^{2R} \nu_N(x_i)^{2(1-R)} + \pi_M(x_i)^{2R} \pi_N(x_i)^{2(1-R)}\Big)^{\frac{1}{R}} \Big]; \qquad (6.3.2)$$

where $R, S > 0$; either $0 < S < 1$ and $1 < R < \infty$ or $0 < R < 1$ and $1 < S < \infty$.

MCDM Approach Based on (R, S)-Norm Pythagorean Fuzzy Information:

Consider a multi-criteria decision-making problem, where $A = \{A_1,\ A_2,\ \ldots,\ A_m\}$ be the set of available alternatives and $C = C_1,\ C_2,\ \ldots,\ C_n$ be the set of criteria. Suppose there is a group of decision makers $D = \{D_1, D_2, \ldots, D_l\}$ who give their opinions and decisions on each alternative with respect to each criterion in the form of linguistic variables. Let $R_k = (h_{ij}^k)$, $i = 1, 2,\ \ldots,\ m$ and $j = 1,\ 2,\ \ldots,\ n$ be the linguistic matrix provided by the each decision maker, say k^{th} decision maker, where h_{ij}^k presents the assessment of an alternative, say A_i, w.r.t. the criterion, say C_j, in terms of linguistic variables.

In order to select the optimal and the best alternative out of the m available alternatives, we devise a modified algorithm based on VIKOR and TOPSIS method by utilizing the Pythagorean fuzzy information measures. For the sake of illustrating the proposed algorithm, we present all the steps through a flow chart given in Figure 6.2 which consists of two stages.

The essential procedural steps for a better understating of the proposed algorithm based on the (R, S)-Norm Pythagorean fuzzy information measures in the VIKOR and TOPSIS methods are listed as follows:

- **Step 1: Assessment of the Criteria by Decision Makers**

  Based upon the experiences, the decision maker gives their individual opinion regarding each criterion utilizing the defined set of linguistic terms connected with the Pythagorean fuzzy number.

- **Step 2: Evaluation of the Decision Maker's Weights**

  It is believed that determining the decision maker's weights is an important concern in a decision-making problem. We assume that the major degree of the decision maker's is obtained by the defined set of linguistic variables and is then written in the form of Pythagorean fuzzy numbers. The weight of k^{th} expert is computed by the following formula [42]:

$$\lambda_k = \frac{\mu_k^2 + \pi_k^2\left[\frac{\mu_k^2}{\mu_k^2+\nu_k^2}\right]}{\sum\limits_{k=1}^{l} \mu_k^2 + \pi_k^2\left[\frac{\mu_k^2}{\mu_k^2+\nu_k^2}\right]}; \tag{6.3.3}$$

  where $\sum\limits_{k=1}^{l} \lambda = 1$ and $\lambda \geq 0$.

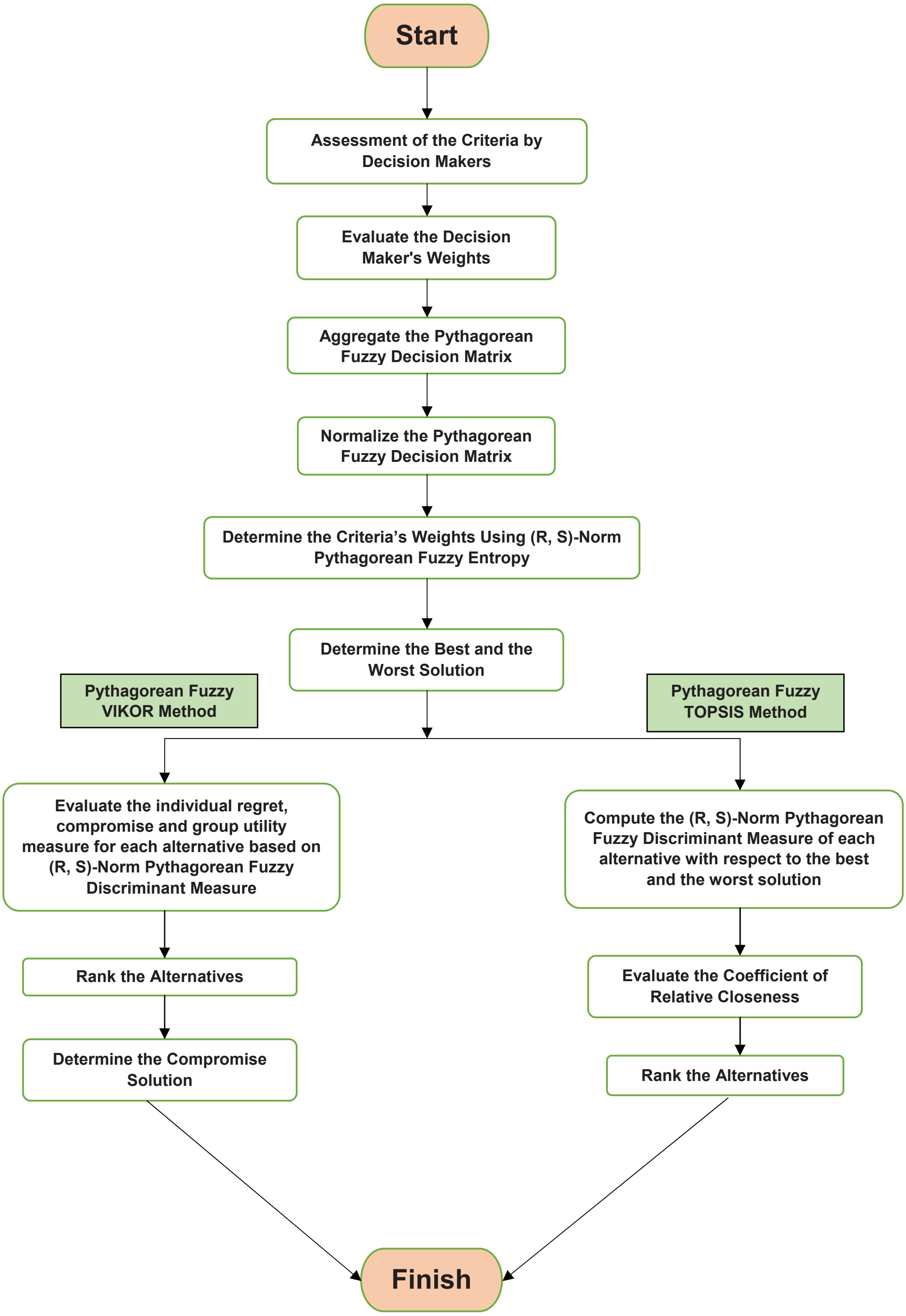

Figure 6.2: Flow Chart of the Proposed Methods

- **Step 3: Determining Aggregated Pythagorean Fuzzy Decision Matrix**

For combining all the individual decision matrices into one group based on decision maker's opinion, we use an averaging aggregation operator to construct the aggregated decision matrix. The following Pythagorean fuzzy weighted averaging operator has been utilized (developed by Yager [107]):

$\tilde{R} = [(\tilde{r}_{ij})]_{m\times n}$, where $\tilde{r}_{ij}$ is

$$\tilde{r}_{ij} = PFWA_{\lambda}(h_{ij}^{(1)}, h_{ij}^{(2)}, \ldots, h_{ij}^{(l)}) = \left(\sqrt{1 - \prod_{k=1}^{l}(1-\mu_{ij}^2)^{\lambda_k}}, \prod_{k=1}^{l}(\nu_{ij})^{\lambda_k}\right) \tag{6.3.4}$$

- **Step 4: Normalization of Pythagorean Fuzzy Decision Matrix**

Sometimes, it has been observed that there is a kind of heterogeneity present in the criterions. For resolving this issue, it is required to make them homogeneous before applying them for any methodology. In a broader sense, the criteria may be categorized into two types: benefit criteria and cost criteria. We transform the decision matrix by transforming the cost criteria into the benefit criteria. Thus the decision matrix $\tilde{R} = [\tilde{r}_{ij}]_{m\times n}$ is converted into a new decision matrix, say, $R = [r_{ij}]_{m\times n}$ where r_{ij} is given by

$$r_{ij} = (\mu_{ij}, \nu_{ij}) = \begin{cases} \tilde{r}_{ij}, & \text{for benefits criteria ;} \\ \tilde{r}_{ij}^{c}, & \text{for cost criteria.} \end{cases} \tag{6.3.5}$$

- **Step 5: Determining the Criteria's Weights**

It may be noted that considering different criteria weights will put an impact in the ranking order of the alternatives. Hence, in the proposed approach we determine the criteria weights by using the (R, S)-Norm Pythagorean fuzzy information entropy as follows:

$$w_j = \frac{1 - e_j}{n - \sum\limits_{j=1}^{n} e_j}, \; j = 1, 2, \ldots, n; \tag{6.3.6}$$

where $e_j = \frac{1}{m}\sum\limits_{i=1}^{m} H_R^S(z_{ij})$, and

$$H_R^S(z_{ij}) = \frac{R\times S}{(R-S)}\sum_{i=1}^{m}\frac{1}{m}\left[\begin{array}{r} \left(\mu_M(x_i)^{2S} + \nu_M(x_i)^{2S} + \pi_M(x_i)^{2S}\right)^{\frac{1}{S}} \\ -\left(\mu_M(x_i)^{2R} + \nu_M(x_i)^{2R} + \pi_M(x_i)^{2R}\right)^{\frac{1}{R}} \end{array}\right].$$

- **Step 6: Identification of the Best and the Worst Solution**

It is essential to determine the best and the worst solution for all the criteria. In the proposed approach, the best and the worst ratings are determined in the form of Pythagorean fuzzy positive ideal solution r_j^+ and Pythagorean fuzzy negative ideal solution r_j^-, which are computed as follows:

$$r_j^+ = \begin{cases} \max\limits_i \mu_{ij}, & \text{for benefit criterion } C_j, \\ \min\limits_i \nu_{ij}, & \text{for cost criterion } C_j; \end{cases} \tag{6.3.7}$$

and

$$r_j^- = \begin{cases} \min\limits_i \mu_{ij}, & \text{for benefit criterion } C_j, \\ \max\limits_i \nu_{ij}, & \text{for cost criterion } C_j. \end{cases} \tag{6.3.8}$$

Remarks: The above stated six steps are the common steps in the stage 1. Further, in stage 2, we may either go for Pythagorean Fuzzy VIKOR method or Pythagorean Fuzzy TOPSIS method depending on the choice of the competent authority. Their respective steps have been listed in two parts as follows:

— Pythagorean Fuzzy VIKOR Method

The VIKOR is one of the important methodology of MCDM introduced by Opricovic [118] to solve decision problems with conflicting criteria with assumption that compromise is acceptable. In literature, this method is one of the widely used MCDM methods for obtaining the compromise solution(s) of the satisfying all the incompatible criteria at the same time. In continuation with the calculations of the six steps stated above, we carry out further calculations to accomplish the decision-making task as follows:

- **Step 7: Evaluation of the Essential Measures for all the alternatives**

In this step we calculate, the essential measures - group utility S_i, individual regret U_i & compromise measure Q_i of every alternative A_i by using the notion of (R, S)-Norm Pythagorean fuzzy discriminant measure. In order to determine the values of these measures of the alternatives A_i $(i = 1, 2, \ldots, m)$, we use the following formula:

$$S_i = \sum_{j=1}^{n} w_j \frac{I_R^S\left(r_j^+, r_{ij}\right)}{I_R^S\left(r_j^+, r_j^-\right)}; \tag{6.3.9}$$

$$U_i = \max_{1\le j\le n} w_j \frac{I_R^S\left(r_j^+, r_{ij}\right)}{I_R^S\left(r_j^+, r_j^-\right)}; \tag{6.3.10}$$

and

$$Q_i = \gamma \frac{\left(S_i - \min\limits_i S_i\right)}{\left(\max\limits_i S_i - \min\limits_i S_i\right)} + (1-\gamma) \frac{\left(U_i - \min\limits_i U_i\right)}{\left(\max\limits_i U_i - \min\limits_i U_i\right)}; \tag{6.3.11}$$

where γ and $1-\gamma$ denote the weights of the strategy of maximum group utility and the weight of the individual regret respectively.

- **Step 8: Ranking of the Alternatives**

  We rank the alternatives based on the decreasing order values of S_i, U_i, Q_i, i.e., the minimum value of the compromise measure Q_i gives the best alternative.

- **Step 9: Determining the Compromise Solution**

  For the uniqueness of the best solution, the alternatives must hold following conditions:

  – **Condition** C_1

  $$Q(A^{(2)}) - Q(A^{(1)}) \geq \frac{1}{m-1}, \tag{6.3.12}$$

  given $A^{(1)}$ is the best ranked alternative and $A^{(2)}$ is the second best ranked alternative by the measure of Q.

  – **Condition** C_2

  $A^{(1)}$ must be the best ranked by S_i or/and U_i. The compromise solution is stable with in a decision-making process, which could be the strategy of maximum group utility (when $\gamma > 0.5$) or by consensus ($\gamma > 0.5$) or with veto ($\gamma < 0.5$).

  In case, if the condition C_1 is not satisfied, then the utmost value of M must be examined and given by the following relation:

  $$Q(A^{(M)}) - Q(A^{(1)}) \leq \frac{1}{m-1};$$

  where M is the arbitrary ranking position of the alternatives other than the best one. As a consequence, the alternative $A^{(i)}$ is the compromise solution for some $i = 1, 2, \ldots, m$.

  — Pythagorean Fuzzy TOPSIS Method

  Hwang and Yoon [21] developed the "*Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS)*" for multi-criteria decision analysis which has been widely used in the literature. The schematic concept behind this method is to choose an alternative which has the shortest geometric distance from the positive ideal solution (PIS) and the longest geometric distance from the negative ideal solution (NIS).

  In continuation with the six steps stated above in the stage 1, we carry out further calculations to accomplish the decision-making task as follows:

- **Step 7: Computation of the (R, S)-Norm Discriminant Measure**

  In this step, we determine the value of the discriminant measures of A_i's $\forall\ i = 1, 2, \ldots, m$ from r_j^+ and r_j^- respectively by taking the discriminant measure (6.3.2) into account.

- **Step 8: Evaluation of the Coefficient of Relative Closeness**

  Evaluate the coefficient of relative closeness, i.e, RCC_i's , $(i = 1, 2, \ldots m)$ as

$$RCC_i = \frac{I_R^S(A_i, r_j^-)}{I_R^S(A_i, r_j^+) + I_R^S(A_i, r_j^-)}. \tag{6.3.13}$$

- **Step 9. Ranking of the Alternatives**

  Finally, we rank the alternatives by ordering the values of the coefficient of relative closeness. The highest value is the best alternative.

Hence, we completely presented the proposed work of solving the decision-making problem in a modified format of VIKOR and TOPSIS by utilizing the (R, S)-Norm Pythagorean fuzzy information measures.

6.4 Hydrogen Power Plant Site Selection Process

In this section, we implement (R, S)-Norm Pythagorean fuzzy information measures in the VIKOR and TOPSIS MCDM methods to obtain the modified form for the hydrogen power plant site selection. The sites under consideration must have been chosen through professional communication by the competent experts. All the criteria affecting the site selection have been determined on the basis of the expert/decision maker's opinion and the available literature. For the sake of selecting the best site/location, the decision makers must take the social aspects, environment aspects, technology aspects, financial implications and also some major characteristic aspects. Consider a selection problem in a conventional frame in which we have four available sites, say, L_1, L_2, L_3 & L_4, which are under consideration in solving the problem. These sites have been systematically examined w.r.t. the five main criteria and 14 sub-criteria (Refer Table 6.1).

It is quite probable that if we increase the number of criteria then we would get a better solution. The problem of site selection may be handled in a more critical way by the experts in a Pythagorean fuzzy set up of VIKOR and TOPSIS technique.

Procedural Steps of Solving the Selection Problem:

Table 6.1: Criteria Affecting the Hydrogen Power Plant Site Selection

Main Criteria	**Sub-criteria**	**Literature Review**
Social Aspect	Public acceptance (F_1)	[58]
	Protection law (F_2)	[22]
	Legal and Regulation compliance (F_3)	[22][58]
Environment Aspect	Availability of Water (F_4)	[126]
	Water Storage (F_5)	[126]
	Environment Affect (F_6)	[126]
Technology Aspect	Distance from Major Road (F_7)	[120] [123] [80] [77]
	Distance from Power Network (F_8)	[23] [120] [123] [80] [77] [74]
	Potential Demand (F_9)	[22] [120]
Economical Aspect	Construction Cost (F_{10})	[22] [80]
	Operation and Management Cost (F_{11})	[22] [58] [123]
	New Feeder Cost (F_{12})	[123]
Site Characteristics	Land Use (F_{13})	[22] [37]
	Ecology (F_{14})	[22]

Table 6.2: Values of Linguistic Terms

Linguistic Term	**PFNs**
Extremely Qualified (EQ)	(0.97, 0.20)
Very Qualified (VQ)	(0.85, 0.35)
Qualified (Q)	(0.55, 0.50)
Less Qualified (LQ)	(0.30, 0.80)
Very Less Qualified (VLQ)	(0.15, 0.90)

- **Step 1.** The linguistic evaluations for the 14 criteria under consideration are qualitatively stated by the decision makers (Ref Table 6.5) and have been transformed into Pythagorean fuzzy information using their quantitative rating in PFNs scale given in Table 6.2. Also, the decision makers provide the qualitative information for four hydrogen power plant sites $L_1, L_2, L_3 \,\&\, L_4$ w.r.t. the 14 criteria (Refer Table 6.4) which have been transformed into Pythagorean fuzzy information by using the defined quantitative rating in PFNs scale given in Table 6.3.

- **Step 2.** In this step, we first present the importance of the decision makers using the linguistic terms which are being transformed into Pythagorean fuzzy information with the defined quantitative rating in PFNs scale given in Table 6.2. Next, we calculate the decision maker's weights using equation (6.3.3) which are being tabulated in Table 6.6.

- **Step 3.** By utilizing the Pythagorean fuzzy weighted averaging aggregation operator given in equation (6.3.4), we aggregate all the decision matrices obtained from the different decision makers to form a single decision matrix. The aggregated matrix hence

Table 6.3: Linguistic Terms for Rating Alternative

Linguistic Term	**PFNs**
Excellently Good (EG)	(0.97, 0.20)
Very Very Good (VVG)	(0.88, 0.30)
Very Good (VG)	(0.80, 0.40)
Good (G)	(0.70, 0.45)
Moderately Good (MG)	(0.65, 0.50)
Fair (F)	(0.55, 0.55)
Moderately Bad (MB)	(0.50,0.65)
Bad (B)	(0.35, 0.80)
Very Bad (VB)	(0.25, 0.88)
Very Very Bad (VVB)	(0.15,0.95)

Table 6.4: Linguistic Evaluation of the Alternatives

		F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
DM1	L_1	EG	EG	MG	EG	EG	EG	VG	G	VG	VG	VG	VG	EG	VVG
	L_2	VG	VG	MG	VG	G	G	MG	G	MG	MG	G	G	VG	VG
	L_3	VG	VG	EG	MG	MG	MG	MG	MG	G	F	F	G	F	F
	L_4	VG	VG	G	VG	G	VG	G	G	MG	EG	VG	VG	VVG	VG
DM2	L_1	EG	EG	MG	VVG	VVG	VVG	G	VVG	VVG	VVG	EG	MG	VVG	VVG
	L_2	MG	G	MG	G	G	G	G	G	G	G	G	VG	VG	VG
	L_3	VG	VVG	VG	MG	F	MG	F	MB	MG	VB	MB	VG	MB	MG
	L_4	VG	VG	G	G	VG	VG	G	MG	MG	EG	VG	G	VG	VG
DM3	L_1	EG	EG	MG	EG	VVG	EG	G	VG	EG	VG	VVG	MG	VG	VVG
	L_2	G	VG	MG	VG	G	MG	F	MG	VG	MG	G	G	VG	VG
	L_3	VG	MB	VVG	MB	F	MG	F	MB	MG	VB	MB	VG	MB	MG
	L_4	MG	MG	VG	VG	G	G	F	MG	VG	MG	G	G	G	VG
DM4	L_1	VVG	VVG	F	VVG	VG	VG	MG	G	VG	G	VG	G	VG	VG
	L_2	MG	G	F	G	MG	F	F	F	G	F	MG	G	G	G
	L_3	G	F	VG	B	F	F	F	B	F	VVB	B	G	B	F
	L_4	F	F	G	G	MG	MG	F	F	G	F	MG	G	MG	G

Table 6.5: Linguistic Evaluation for Rating Criteria

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
DM1	VQ	VQ	Q	EQ	EQ	VQ	Q	Q	VQ	EQ	VQ	VQ	EQ	EQ
DM2	VQ	EQ	EQ	VQ	EQ	EQ	LQ	VQ	Q	VQ	LQ	VQ	EQ	Q
DM3	EQ	Q	VQ	EQ	LQ	VQ	VQ	VQ	EQ	Q	VQ	VQ	Q	EQ
DM4	EQ	LQ	Q	EQ	Q	Q	VQ	Q	EQ	LQ	Q	Q	Q	VQ

Table 6.6: Decision Maker's Weights

	DM1	**DM2**	**DM3**	**DM4**
Linguistic Term	VQ	Q	EQ	VQ
Weight	0.265802	0.170204	0.298192	0.265802

Table 6.7: Aggregated Pythagorean Fuzzy Decision Matrix

	L_1	L_2	L_3	L_4
F_1	(0.957, 0.223)	(0.714, 0.457)	(0.778, 0.413)	(0.711, 0.465)
F_2	(0.957, 0.223)	(0.762, 0.421)	(0.712, 0.479)	(0.711, 0.465)
F_3	(0.627, 0.513)	(0.627, 0.513)	(0.898, 0.305)	(0.735, 0.434)
F_4	(0.946, 0.239)	(0.762, 0.421)	(0.550, 0.613)	(0.762, 0.421)
F_5	(0.906, 0.291)	(0.688, 0.463)	(0.598, 0.528)	(0.709, 0.454)
F_6	(0.938, 0.258)	(0.652, 0.490)	(0.611, 0.521)	(0.740, 0.440)
F_7	(0.449, 0.721)	(0.518, 0.609)	(0.518, 0.609)	(0.504, 0.626)
F_8	(0.405, 0.775)	(0.490, 0.652)	(0.613, 0.550)	(0.499, 0.642)
F_9	(0.310, 0.898)	(0.447, 0.724)	(0.507, 0.627)	(0.455, 0.717)
F_{10}	(0.393, 0.797)	(0.504, 0.637)	(0.720, 0.439)	(0.344, 0.880)
F_{11}	(0.326, 0.877)	(0.463, 0.688)	(0.639, 0.492)	(0.440, 0.740)
F_{12}	(0.712, 0.458)	(0.721, 0.441)	(0.775, 0.405)	(0.732, 0.436)
F_{13}	(0.891, 0.317)	(0.778, 0.413)	(0.516, 0.628)	(0.775, 0.407)
F_{14}	(0.863, 0.324)	(0.778, 0.413)	(0.601, 0.526)	(0.769, 0.424)

obtained is presented in Table 6.7.

- **Step 4.** In this step, we normalize the obtained aggregated Pythagorean fuzzy decision matrix using the equation (6.3.5) and the computed normalized Pythagorean matrix is presented in Table 6.8.

- **Step 5.** In this step, the weights of all the criteria have been evaluated using the (R, S)-Norm Pythagorean fuzzy entropy measure given by equation (6.3.1) and the computed values of criteria's weights are tabulated in Table 6.9.

- **Step 6.** The computed values of Pythagorean fuzzy positive ideal solution r_j^+ and Pythagorean fuzzy negative ideal solution r_j^- are as follows:

$$\begin{aligned} r_j^+ = \{&(0.957, 0.223), (0.957, 0.223), (0.898, 0.305), (0.946, 0.239), (0.906, 0.291), (0.938, 0.258),\\ &(0.449, 0.721), (0.405, 0.775), (0.310, 0.898), (0.344, 0.880), (0.326, 0.877), (0.712, 0.458),\\ &(0.891, 0.317), (0.863, 0.324)\}; \end{aligned} \tag{6.4.1}$$

and

$$\begin{aligned} r_j^- = \{&(0.711, 0.465), (0.711, 0.479), (0.627, 0.513), (0.550, 0.613), (0.598, 0.528), (0.611, 0.521),\\ &(0.518, 0.609), (0.613, 0.550), (0.507, 0.627), (0.720, 0.439), (0.639, 0.492), (0.775, 0.405),\\ &(0.516, 0.628), (0.601, 0.526)\}. \end{aligned} \tag{6.4.2}$$

Remark: The above stated steps comprise of all the six common steps of the proposed methodology. Next, we first carry the steps in connection with the Pythagorean fuzzy VIKOR method and then with the Pythagorean fuzzy TOPSIS method.

Table 6.8: Normalized Aggregated Pythagorean Fuzzy Decision Matrix

	L_1	L_2	L_3	L_4
F_1	(0.957, 0.223)	(0.714, 0.457)	(0.778, 0.413)	(0.711, 0.465)
F_2	(0.957, 0.223)	(0.762, 0.421)	(0.712, 0.479)	(0.711, 0.465)
F_3	(0.627, 0.513)	(0.627, 0.513)	(0.898, 0.305)	(0.735, 0.434)
F_4	(0.946, 0.239)	(0.762, 0.421)	(0.550, 0.613)	(0.762, 0.421)
F_5	(0.906, 0.291)	(0.688, 0.463)	(0.598, 0.528)	(0.709, 0.454)
F_6	(0.938, 0.258)	(0.652, 0.490)	(0.611, 0.521)	(0.740, 0.440)
F_7	(0.721, 0.449)	(0.609, 0.518,)	(0.609, 0.518)	(0.626, 0.504)
F_8	(0.775, 0.405)	(0.652, 0.490)	(0.550, 0.613)	(0.642, 0.499)
F_9	(0.898, 0.310)	(0.724, 0.447)	(0.627, 0.507)	(0.717, 0.455)
F_{10}	(0.797, 0.393)	(0.637, 0.504,)	(0.439, 0.720)	(0.880, 0.344,)
F_{11}	(0.877, 0.326)	(0.688, 0.463)	(0.492, 0.639)	(0.740, 0.440)
F_{12}	(0.458, 0.712)	(0.441, 0.721)	(0.405, 0.775)	(0.436, 0.732)
F_{13}	(0.891, 0.317)	(0.778, 0.413)	(0.516, 0.628)	(0.775, 0.407)
F_{14}	(0.863, 0.324)	(0.778, 0.413)	(0.601, 0.526)	(0.769, 0.424)

Table 6.9: Evaluation of Criteria's Weights

Criteria	**Weights** (w_i)
F_1	0.1070
F_3	0.0514
F_4	0.1339
F_5	0.0900
F_6	0.0657
F_7	0.0361
F_8	0.0244
F_9	0.0926
F_{10}	0.0900
F_{11}	0.0361
F_{12}	0.0387
F_{13}	0.0783
F_{14}	0.0926

Table 6.10: Computation Outcomes and Compromise Measure of Each Site

	S_i	U_i	Q_i
L_1	0.129	0.073	0.000
L_2	1.069	0.295	0.8589
L_3	1.437	0.233	0.8596
L_4	0.952	0.293	0.8091
Ranking Order	$S_1 > S_4 > S_2 > S_3$	$U_1 > U_3 > U_4 > U_2$	$Q_1 > Q_4 > Q_2 > Q_3$

— Pythagorean Fuzzy VIKOR Method

- **Step 7.** Using equations (6.3.9), (6.3.10) and (6.3.11), we determine the values of S_i, U_i and Q_i respectively. For calculating the values of the compromise measure, we take $\gamma = 0.5$. The computed values are tabulated in the Table 6.10.

- **Step 8.** On the basis of the computed values of S_i, U_i and Q_i in the step 7, the ranking results have been obtained as follows:

$$S_1 > S_4 > S_2 > S_3;\ U_1 > U_3 > U_4 > U_2;\ Q_1 > Q_4 > Q_2 > Q_3.$$

- **Step 9.** Based on the descending order of the obtained values of the Q_i's, the site L_1 is supposed to be the best appropriate site. Since

$$Q(A^{(2)}) - Q(A^{(1)}) = 0.8091 > \frac{1}{4-1} = 0.333,$$

therefore, the site L_1 also fulfill the condition C_1 & C_2. Thus, we jointly conclude that the site L_1 is the most suitable location to setup a hydrogen power plant.

Sensitivity Analysis of the Obtained Solution:

In order to observe the changes in the ranking order for different suppositions of the weights (γ ($0 \leq \gamma \leq 1$)) of the strategy of maximum group utility, we carry out a sensitivity analysis for the compromise solution as shown in Table 6.11, Figure 6.3 and Figure 6.4. In view of the obtained values in the Table 6.11, we conclude that the site L_1 is the most suitable location for setting up the hydrogen power plant which can also be viewed in Figure 6.5.

Table 6.11: Sensitivity Analysis for Different Values of γ

	L_1	L_2	L_3	L_4
S_i	0.129	1.069	1.437	0.952
U_i	0.073	0.295	0.233	0.293
Q_i ($\gamma = 0.0$)	0.0	1.0	0.719293091	0.989211567
Q_i ($\gamma = 0.1$)	0.0	0.97179769	0.747363782	0.953178618
Q_i ($\gamma = 0.2$)	0.0	0.94359538	0.775434472	0.917145668
Q_i ($\gamma = 0.3$)	0.0	0.91539307	0.803505163	0.881112718
Q_i ($\gamma = 0.4$)	0.0	0.88719076	0.831575854	0.845079768
Q_i ($\gamma = 0.5$)	0.0	0.85898845	0.859646545	0.809046819
Q_i ($\gamma = 0.6$)	0.0	0.83078614	0.887717236	0.773013869
Q_i ($\gamma = 0.7$)	0.0	0.80258383	0.915787927	0.736980919
Q_i ($\gamma = 0.8$)	0.0	0.77438152	0.943858618	0.700947969
Q_i ($\gamma = 0.9$)	0.0	0.74617921	0.971929309	0.66491502
Q_i ($\gamma = 1.0$)	0.0	0.7179769	1.0	0.62888207

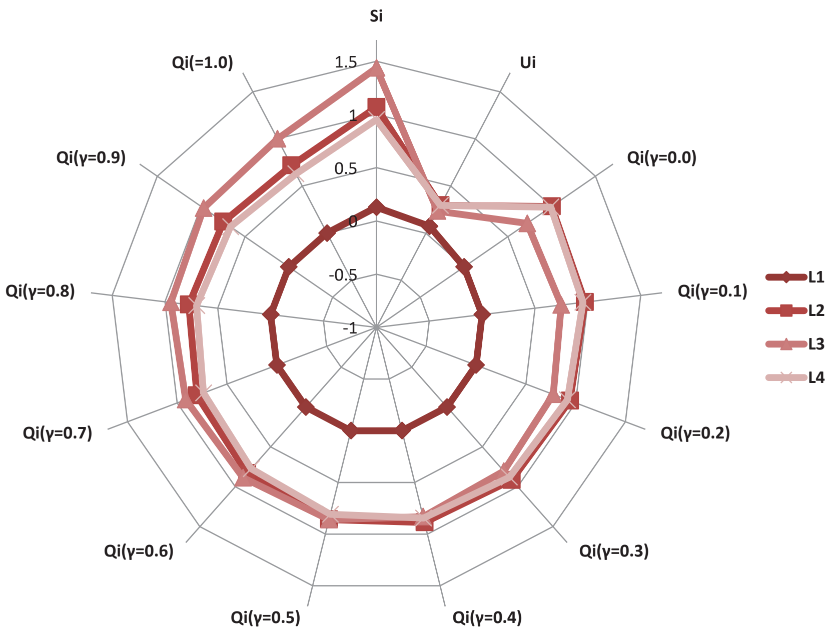

Figure 6.3: Sensitivity Study of Alternatives w.r.t. Measures

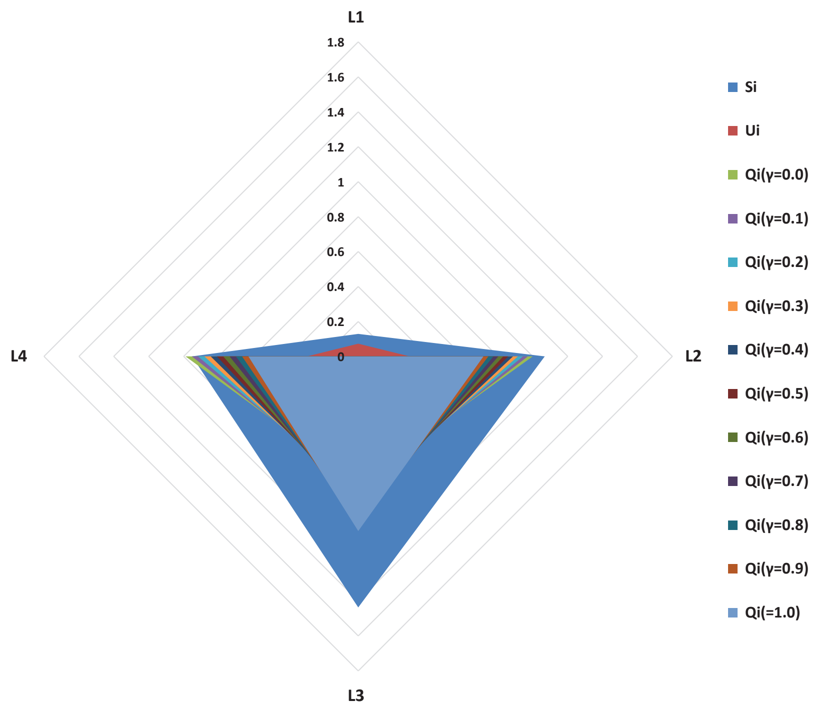

Figure 6.4: Sensitivity Study of Compromise Measure

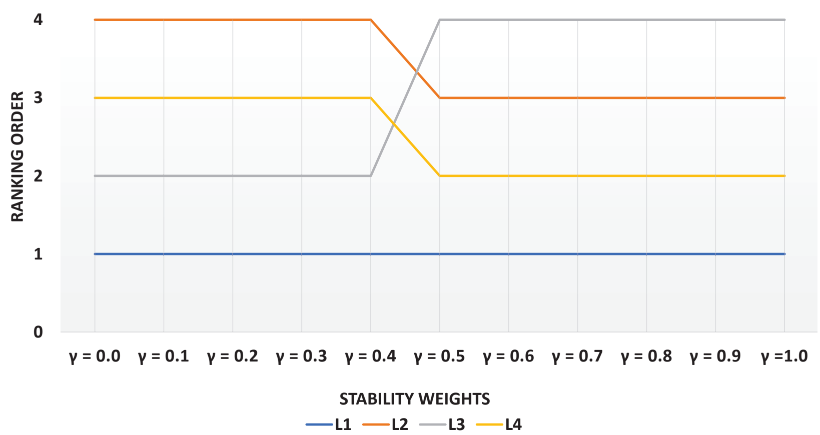

Figure 6.5: Ranking Order w.r.t. Stability Weights (γ)

— **Pythagorean Fuzzy TOPSIS Method**

- **Step 7.** In this step, we evaluate the values of the discriminant measures of L_i's $\forall$ $i = 1, 2, 3, 4$ from r_j^+ and r_j^- respectively with the help of the equation (6.3.2) and presented in Table 6.12.

Table 6.12: Discriminant Measure for $L_i^{'}s$ w.r.t. r_j^+/r_j^-

	$I_R^S(L_i, r_j^+)$	$I_R^S(L_i, r_j^-)$
L_1	0.3349	0.495
L_2	0.5398	0.358
L_3	0.6342	0.324
L_4	0.5036	0.389

- **Step 8.** We compute the values of the coefficient of relative closeness by using the equation (6.3.13) and put in Table 6.13.

Table 6.13: Coefficient of Relative Closeness

Sites	**Closeness Index**
L_1	0.5967
L_2	0.3986
L_3	0.3381
L_4	0.4360

- **Step 9.** On the basis of the computed values of the coefficient of relative closeness, the ranking results have been obtained as follows:

$$L_1 > L_4 > L_2 > L_3.$$

Hence, based on the coefficient of closeness, the site L_1 is the most suitable location to setup a hydrogen power plant.

Remark: It may be observed that the ranking results obtained through both the methodologies, i.e., Pythagorean Fuzzy VIKOR and TOPSIS MCDM methods, are completely consistent and acceptable.

6.5 Comparative Analysis and Advantages

In this section, a comparative analysis by taking the results of the proposed methodologies and various other existing methods into account has been carried out to illustrate the advantages of the proposed methodologies - modified Pythagorean fuzzy VIKOR and TOPSIS MCDM techniques. The following are the important comparative remarks and advantages:

- The risk information loss has been significantly minimized in the proposed methods as the computations consider the (R, S)-Norm Pythagorean entropy measure and discriminant measure which spans a wider information in the fulfilment of the criteria. The incorporation of the parameters enables us to have the flexibility in the calculations along with the family of information measures.

- The proposed modified VIKOR and TOPSIS method incorporate the notion of Pythagorean fuzzy sets while various researchers have implemented FSs/IFSs which are the special case of Pythagorean fuzzy sets. As discussed in the introduction, Pythagorean fuzzy sets are more generalized and have a wider coverage for the imprecise and incomplete information.

- We have appropriately assigned the weights to the expert's/decision maker's opinion in developing the proposed methodologies which provide the more precise decisions for the MCDM problems under consideration while Boran et al. [42] utilized intuitionistic approach in solving the group decision-making supplier selection problem with TOPSIS method in a straight way.

- One of the major advantage of the Pythagorean fuzzy VIKOR approach is that it yields the compromise solution which takes the maximum group utility along with the minimum individual regret. The compromise solution obtained though the modified VIKOR method is the best solution with respect to the ideal solution.

- The notion of Pythagorean fuzzy numbers have the capability to deal with the imprecise and incomplete information which arise in a MCDM problem. Since the input parameters - assessment of alternatives, decision maker's weights and the criteria's weights may have uncertainty in the content, therefore, the implementation of the notion of Pythagorean fuzzy number is found to be more appropriate.

- It may be noted that a MCDM method broadly consists of different essential characteristics, viz., weights of criteria, expert's/decision maker's weights and evaluation of available alternatives with the laid down criteria. Therefore, any novel approach in the field of MCDM focuses on these stated characteristics. Here, we present the comparison

Table 6.14: Comparison with the Various Existing Methods

Authors & Researchers	**Expert Weight**	**Criteria Weight**	**Linguistic terms**	**Entropy and Discriminant Measure**	**Alternative Assessment Information**
Kaya & Kahraman [124]	Consider	Partially Known	Yes	NO	FS
Kahraman & Kaya [18]	Consider	Partially Known	Yes	NO	FS
Mousavi et al. [78]	Computed	Completely Unknown	Yes	NO	HFS
Mishra et al. [9]	Consider	Partially Known	Yes	Discriminant Measure	IFS
Schitea et al. [36]	Computed	Completely Unknown	Yes	NO	IFS
Proposed Work	Computed	Completely Unknown	Yes	Both	PFS

Table 6.15: Comparison with the Various Existing Methods

	WASPAS [36]	COPRAS [36]	EDAS [36]	Proposed VIKOR	Proposed TPOSIS
L_1	1	1	1	1	1
L_2	3	3	3	3	3
L_3	4	4	4	4	4
L_4	2	2	2	2	2

of our proposed research based on these features with various existing approaches in literature along with its advantages as shown in Table 6.14.

- The final ranking for the available hydrogen power plant site alternatives studied by various researchers in recent past is summarized and tabulated in Table 6.15 which shows a clear and crisp consistency of the proposed methodologies. It may be observed that the results obtained are statistically similar but the proposed methods are different from the other methods available in literature.

- The utilization of the method is supposed to coordinate with the requirement of decision-making environment. VIKOR can maximize the group utility and minimize the individual regret while TOPSIS is a compensatory method which allows the trade-off between criteria. With respect to our proposed methods, it can consider the advantage aspects of both VIKOR and TOPSIS. In general, our proposed methods make full use of more information during the decision-making procedure, which are more suitable for a complex environment.

6.6 Conclusion

In this chapter, a new parametric (R, S)-Norm Pythagorean fuzzy entropy measure and respective Pythagorean fuzzy discriminant measure have been successfully incorporated in the VIKOR and TOPSIS MCDM methods to propose the modified MCDM methodologies which is more generalized in its own sense. The utilized Pythagorean fuzzy information measures have been found to be significantly efficient to handle the uncertainty where the weights of the criteria are completely unknown. In addition, the respective sensitivity analysis has also been done for the sake of better understanding and readability. The literature review clearly shows the novelty of the proposed approach. The 14 criteria used for the hydrogen power plant site selection give a comprehensive coverage for the experts/decision makers. The detailed comparative study clearly shows that we have obtained a completely feasible and equally consistent ranking which are in a more general frame work of Pythagorean fuzzy information.

Chapter 7

Conclusions

In the present thesis, we have studied and proposed some new decision making approaches along with various results for different problems under Pythagorean fuzzy setup. For the sake of presenting the concluding remarks of the thesis, we are listing the findings of the work carried out in various chapters as follows:

- The notion of Pythagorean fuzzy soft matrix (PFSM) has been successfully introduced with different categories, properties & various standard binary operations.
- The general structured decision making problem has been illustrated and solved with the help of revised definition of choice/weighted choice matrix.
- A new approach, by taking Pythagorean fuzzy soft matrix into consideration, for solving a general medical diagnosis problem has been well presented by incorporating the score/utility matrix.
- A comparison analysis is also carried to show the practicability and consistency of the proposed algorithms with the help of numerical examples and the existing literature.
- In continuation, a new technique for the dimensionality reduction of the informational data has been presented in the Pythagorean fuzzy setup.
- A new methodology for the dimensionality reduction based on the reframed object-oriented/parameter-oriented PFSMs has been successfully presented with comparative analysis. Consequently, the consistency and viability of the proposed algorithm in contrast with methodologies available in literature have been duly discussed.

- Further, a new kind of parametric information measure, termed as (R, S)-norm Pythagorean fuzzy entropy measure has been presented with proper validation. Some important properties, monotonicity and maximality of proposed entropy measure have also been studied with due consideration.

- A new approach to determine the weight of criteria for two different cases (weights partially known or weights completely unknown) has been proposed to present a methodology for solving the general structured multi-criteria decision making problem. A numerical example has also been solved for better understanding.

- Next, a bi-parametric (R, S)-norm Pythagorean fuzzy discriminant measure has been successfully presented with different important properties. The monotonic nature of the discriminant measure has also been studied empirically for necessary validation.

- Some new approaches based on the proposed parametric discriminant measure for solving different types of soft computing problem have been discussed and each computational application has also been illustrated with the help of an illustrative example.

- Upon utilizing the Pythagorean fuzzy (R, S)-norm entropy measure and the (R, S)-norm divergence measure, the standard VIKOR and TOPSIS approaches for solving the MCDM problem have been accordingly presented and modified.

- The problem of site selection of hydrogen power plant has been remodeled in Pythagorean fuzzy setup and solved with the help of the proposed modified VIKOR and TOPSIS methods. Finally, the practicability and consistency of the proposed method have been studied.

- The various methodologies presented in the thesis can further be applied on the real survey data of the real world decision making problems and results may be derived based on the necessary modeling and simulation.

- The proposed dimensionality reduction technique may further be applied in enhancing the performance of large scale image retrieval.

- In future, the proposed modified VIKOR and TOPSIS MCDM approach can also be used for location selection of different types of renewable energy resources or any other selection problem.

Bibliography

[1] A. A. Konate, H. Pan, H. Ma, X. Cao, Y. Y. Ziggah, M. Oloo, N. Khan, "Application of dimensionality reduction technique to improve geo- physical log data classification performance in crystalline rocks," *Journal of Petroleum Science and Engineering*, vol. 133, pp. 633–645, 2015.

[2] A. Aktas, M. Kabak, "A Hybrid Hesitant Fuzzy Decision-Making Approach for Evaluating Solar Power Plant Location Sites," *Arabian Journal for Science and Engineering*, vol. 44, pp. 7235–7247, 2019.

[3] A. Deluca, S. Termini, "A definition of non-probabilistic entropy in the setting of fuzzy set theory," *Information and control*, vol. 20, pp. 301–312, 1971.

[4] A. G. Hatzimichailidis, G. A.Papakostas, V. G.Kaburlasos, "A novel distance measure of intuitionistic fuzzy sets and its application to pattern recognition problems," *International Journal of Intelligent Systems*, vol. 27(4), pp. 396–409, 2012.

[5] A. Guleria, R. K. Bajaj, "Pythagorean Fuzzy (R, S)-norm Information Measure based Multicriteria Decision Making Problem," *Advances in Fuzzy Systems*, vol. 2018, pp. 1–11, 2018.

[6] A. Guleria, R. K. Bajaj, "On Pythagorean Fuzzy Soft Matrices, Operations and their Applications in Decision Making and Medical Diagnosis," *Soft Computing*, vol. 23(17), pp. 7899–7900, 2019.

[7] A. Guleria, R. K.Bajaj, "Pythagorean Fuzzy(R,S)-Norm Divergence Measure in Various Decision Making Processes," *Journal of Intelligent & Fuzzy Systems*, vol. 38(3), pp. 761–777, 2019.

[8] A. Kaufmann, "Fuzzy subsets: Fundamentals Theorectical Elements Vol. III,“ Academic Press, New York.

[9] A. R. Mishra, R. Kumari, D. K. Sharma, "Intuitionistic fuzzy divergence measure-based multi-criteria decision making method," *Neural Computing and Applications*, vol. 31, pp. 2279–2294, 2019.

[10] A. Renyi, "On measures of entropy and information, Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability, Vol. I," *University of California Press*, Berkeley, Calif., pp. 541–561, 1961.

[11] A. Simonnet, "Technical options for distributed hydrogen refueling stations in a market driven situation," Institute of transportation Studies UC Davis, University of California, Report UCD-ITS-RP-05-08, 2005.

[12] A. T. W. Chu, R. E. Kalaba, K. Spingarn, "A comparison of two methods for determining the weights of belonging to fuzzy sets," Journal of Optimization Theory and Applications, vol. 27, pp. 531–538, 1979.

[13] B. Chetia, P. K. Das, "Some results of Intuitionistic fuzzy soft matrix theory," *Advances in Applied Science Research*, vol. 3, pp. 412–423, 2012.

[14] B. D. Sharma, D. P. Mittal, "New non-additive measures of entropy for discrete probability distributions," *Journal of Mathematical Sciences*, vol. 10, pp. 28–40, 1975.

[15] B. Su, X. Ding, H. Wang, Y. Wu, "Discriminative Dimensionality Reduction for Multi-Dimensional Sequences," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 40(1), pp. 77–91, 2018.

[16] C. E. Shannon, "A Mathematical theory of communication," *Bell System Technical Journal*, vol. 27(3), pp. 379–423, 1948.

[17] C. He, H. Sun, Y. Xu, S. Lv, "Hydrogen refueling station siting of expressway based on the optimization of hydrogen life cycle cost", *International Journal of Hydrogen Energy*, vol. 42(26), pp. 16313–16324, 2017.

[18] C. Kahraman, V. Q. Kaya, "A fuzzy multi-criteria methodology for selection among energy alternatives," *Expert Systems with Applications*, vol. 37(9), pp. 6270–6281, 2010.

[19] C. L. Hwang, M. J. Lin, "Group decision making under multiple criteria: methods and applications," Springer, Berlin, Germany, 1987.

[20] C. N. Wang, C. C. Su, V. T. Nguyen, "Nuclear Power Plant Location Selection in Vietnam under Fuzzy Environment Conditions," *Symmetry*, vol. 11, 548, 2018.

[21] C. L. Hwang, K. P. Yoon, *Multiple Attribute Decision Making Methods and Applications*, Springer, New York, 1981.

[22] C. N. Wang, V. T. Nguyen, H. T. N. Thai, D. H. Duong, “Multi-Criteria Decision Making (MCDM) Approaches for Solar Power Plant Location Selection in Vietnam,” *Energies*, vol. 11, 1504, 2018.

[23] C. N. Wang, V. T. Nguyen, H. T. N. Thai, D. H. Duong, “A Hybrid Fuzzy Analysis Network Process (FANP) and the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) Approaches for Solid Waste to Energy Plant Location Selection in Vietnam,” *Journal of Applied Sciences*, vol. 8, 1100, 2018.

[24] C. Naim, E. Serdar, “Soft matrix theory and its decision making,” *Computers and Mathematics with Applications*, vol. 59, pp. 3308–3314, 2010.

[25] C. P. Wei, Z. H. Gao, T. T. Guo, “An intuitionistic fuzzy entropy measure based on trigonometric function,” Control and Decision, vol. 27(4), pp. 571–574, 2012.

[26] C. Stiller, P. Seydel, U. Munger, M. Wietschel, “Early hydrogen user centers and corridors as a part of the European hydrogen energy roadmap (HyWays),” *International Journal of Hydrogen Energy*, vol. 33, pp. 4193–4209, 2008.

[27] D. A. Molodstov, “Soft set theory-first result,” *Computers and Mathematics with Application*, vol. 27, pp. 19–31, 1999.

[28] D. Bhandari, N. R. Pal, “Some new information measures for fuzzy sets,” *Information Sciences*, vol. 67(3), pp. 204–228, 1993.

[29] D. Chen, E. C. C. Tsang, D. S. Yeungand, X. Wang, “The parameterization reduction of soft sets and its application,” *Computer and Mathematics with Applications*, vol. 49, pp. 757–763, 2005.

[30] D. E. Boekee, J. C. A. Van der Lubbe, “The R-norm information measure,” *Information and Control*, vol. 45(2), pp.136–155, 1980.

[31] D. F. Li, “Some measures of dissimilarity in intuitionistic fuzzy structures,” *Journal of Computer and System Sciences*, vol. 68(1), pp. 115–122, 2004.

[32] D. F. Li, “Multiattribute decision making models and methods using intutionistic fuzzy sets,” *Journal of Computer and System Sciences*, vol. 70(1), pp. 73–85, 2005.

[33] D. Messaoudi, N. Settou, B. Negrou, B. Settou, "GIS based multi-criteria decision making for solar hydrogen production sites selection in Algeria", *International Journal of Hydrogen Energy*, vol. 44(60), pp. 31808–31831, 2019.

[34] D. Pamucar, L. Gigovic, Z. Bajic, M. Janosevic, "Location Selection for Wind Farms Using GIS Multi-Criteria Hybrid Model: An Approach Based on Fuzzy and Rough Numbers," *Sustainability*, vol. 9, 1315, 2017.

[35] D. S. Hooda, R. Kumari, "On Applications of Fuzzy Soft Sets in Dimension Reduction and Medical Diagnosis," *Advances in Research*, vol. 12(2), pp. 1–9, 2017.

[36] D. Schitea, M. Deveci, M. Iordache, K. Bilgili, V. Q. Z. Akyurt, V. Q. Iordache, "Hydrogen Mobility roll-up site selection using intuitionistic fuzzy sets based WASPAS, COPRAS and EDAS," *Internation Journal of Hydrogen Energy*, vol. 44(16), pp. 8585–8600, 2019.

[37] E. Chamanehpour, A. Akbarpour, "Site selection of wind power plant using multi-criteria decision-making methods in GIS: A case study," *Computational Ecology and Software*, vol. 7, pp. 49–64, 2017.

[38] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[39] E. Szmidt, J. Kacprzyk, "Entropy for intuitionistic fuzzy sets," *Fuzzy sets and systems*, vol. 118(3), pp. 467–477, 2001.

[40] E. Szmidt, J. Kacprzyk, "A similarity measure for intuitionistic fuzzy sets and its application in supporting medical diagnostic reasoning, in: *International Conference Artificial Intelligence and Soft Computing*, Zakopane, Poland, pp. 388–393, 2004.

[41] E. U. Choo, W. C. Wedley, "Optimal criterion weights in repetitive multicriteria decision making," *Journal of the Operational Research Society*, vol. 36, pp. 983–992, 1985.

[42] F. E. Boran, S. Genc, M. Kurt, D. Akay, "A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method," *Expert Systems with Applications*, vol. 36(8), pp. 11363–11368, 2009.

[43] F. Sedady, M. A. Beheshtinia, "A novel MCDM model for prioritizing the renewable power plants' construction," *Management of Environmental Quality An International Journal*, vol. 30, pp. 383–399, 2019.

[44] F. Xiao, W. Ding, "Divergence measure of Pythagorean fuzzy sets and its application in medical diagnosis," *Applied Soft Computing*, vol. 79, pp. 254–267, 2019.

[45] G. Anandarajah, E. P. McDowall, "Decarbonizing read transport with hydrogen and electricity: long term global technology learning scenarios," *International Journal of Hydrogen Energy*, vol. 38, pp. 3419–3432, 2013.

[46] G. A. Papakostas, A. G. Hatzimichailidis, V. G. Kaburlasos, "Distance and similarity measures between intuitionistic fuzzy sets, a comparative analysis from a pattern recognition point of view," *Pattern Recognition Letters*, vol. 34(14), pp. 1609–1622, 2013.

[47] G. C. Biswal, S. P. Shukla, "Site Selection for Wind Farm Installation," *International Journal of Innovative Research in Electrical, Electronics, Instrumentation and Control Engineering*, vol. 3, pp. 59–61, 2015.

[48] G. Juste, "Hydrogen injection as additional fuel in gas turbine combustor. Evaluation of effects," *International Journal of Hydrogen Energy*, vol. 31, pp. 2112–2121, 2006.

[49] G. Wei, Y. Wei, "Similarity measures of Pythagorean fuzzy sets based on cosine function and their applications," *International Journal of Intelligent Systems*, vol. 33(3), pp. 634–652, 2018.

[50] G. Wei, "Pythagorean fuzzy interaction aggregation operators and their application to multiple attribute decision making," *Journal of Intelligent and Fuzzy Systems*, vol. 33(4), pp. 2119–2132, 2017.

[51] H. B. Mitchell, "On the Dengfeng-Chuntian similarity measure and its application to pattern recognition," *Pattern Recognition Letters*, vol. 24, pp, 3101–3104, 2013.

[52] H. Garg, "A novel correlation coefficients between Pythagorean fuzzy sets and its applications to decision making processes," *International Journal of Intelligent Systems*, vol. 31(12), pp. 1234–1252, 2016.

[53] I. K. Vlachos, G. D. Sergiadis, "Intuitionistic fuzzy information, Applications to pattern recognition," *Pattern Recognition Letters*, vol. 28, pp. 197–206, 2007.

[54] I. Kaya, C. Kahraman, "A comparison of fuzzy multicriteria decision making methods for intelligent building assessment," *Journal of Civil Engineering and Management*, vol. 20(1), pp. 59–69, 2014.

[55] I. Montes, N. R. Pal, V. Janis, S. Montes, "Divergence Measures for Intuitionistic Fuzzy Sets", *IEEE Transactions on Fuzzy Systems*, vol. 23(2) pp. 444–456, 2015.

[56] I. Mukhametzyanov, D. Pamucar, "A sensitivity analysis in MCDM problems: A statistical approach," *Decision Making: Applications in Management and Engineering*, vol. 1(2), pp. 51–80, 2018.

[57] I. Perfilieva, "Dimensionality Reduction by Fuzzy Transforms with Applications to Mathematical Finance," in Anh L., Dong L., Kreinovich V., Thach N. (eds), *Econometrics for Financial Applications (ECONVN 2018), Studies in Computational Intelligence*, vol. 760, Springer, Cham, 2018.

[58] I. Talinli, E. Topuz, E. Aydin, S.B. Kabakcy, "A Holistic Approach for Wind Farm Site Selection by Using FAHP, Wind Farm-Technical Regulations, Potential Estimation and Siting Assessment," Gaston O. Suvire, IntechOpen, DOI: 10.5772/17311.

[59] J. E. Shore, R. M. Gray, "Minimization cross-entropy pattern clasiification and cluster analysis," *IEEE Transaction Pattern Analysis Machine Intelligence*, vol. 4(1), pp. 11–17, 1982.

[60] J. Fan, W. Xie, "Distance measures and induced fuzzy entropy," *Fuzzy Sets and Systems*, vol. 104(2), pp. 305–314, 1999.

[61] J. Havrda, F. Charvat, "Quantification method of classification processes: concept of structural α-entropy," *Kybernetika*, vol. 3(1), pp. 30–35, 1967.

[62] J. Lewandowska-Smierzchalska, R. Tarkowski, B. Uliasz-Misiak, "Screening and ranking framework for underground hydrogen storage site selection in Poland", *International Journal of Hydrogen Energy*, vol. 43(9), pp. 4401–4414, 2018.

[63] J. P. Brans, V. Mareschel, "PROMETHEE: A new family of outranking methods in multicriteria analysis," *In J. P. Brans (Ed.), Operational research 84*, pp. 477–490, New York: North-Holland, 1984.

[64] J. Wang, P. Wang, "Intuitionistic linguistic fuzzy multi-criteria decision-making method based on intuitionistic fuzzy entropy," *Control and Decision*, vol. 27 pp. 1694–1698, 2012.

[65] J. Ye, "Two effective measures of intuitionistic fuzzy entropy," *Computing*, vol. 87(1-2), pp. 55–62, 2010.

[66] K. K. Borah, S. Roy, T. Harinarayana, "Optimization in Site Selection of Wind Turbine for Energy Using Fuzzy Logic System and GIS– A Case Study for Gujarat," *Open Journal of Optimization*, vol. 2, pp. 116–122, 2013.

[67] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, pp. 87–96, 1986.

[68] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[69] L. A. Zadeh, "Similarity relations and fuzzy orderings," *Information Science*, vol. 3, pp. 177–200, 1971.

[70] L. Dengfeng, C. Chuntian, "New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions," *Pattern Recognition Letters*, vol. 23, pp. 221–225, 2002.

[71] L. F. A. M. Gomes, M. M. P. P. Lima, "Todim: Basic and application to multicriteria ranking of projects with environmental impacts," Foundations of Computing and Decision Sciences, vol. 16, pp. 113–127, 1991.

[72] L. P. Dominguez, L. A. Rodrguez-Picon, A. Alvarado-Iniesta,D. L. Cruz, Z. Xu, "MOORA under Pythagorean Fuzzy Set for Multiple Criteria Decision Making," *Complexity*, vol. 2018, pp. 1–10, 2018.

[73] M. Deveci, "Site selection for hydrogen underground storage using interval type-2 hesitant fuzzy sets", *International Journal of Hydrogen Energy*, vol. 43(19), pp. 9353–9368, 2018.

[74] M. G. Toklu, O. Uygun, "Location Selection for Wind Plant using AHP and Axiomatic Design in Fuzzy Environment," *Periodicals of engineering and Natural Sciences*, vol. 6, pp. 120–128, 2018.

[75] M. Ghosh, D. Das, C. Ray, A. K. Chakraborty, "Autumated leukocyte recoginition using fuzzy divergence," *Micron*, vol. 41(7), pp. 840–846, 2010.

[76] M. Karatas, "Hydrogen energy storage method selection using fuzzy axiomatic design and analytic hierarchy process", *International Journal of Hydrogen Energy*, vol. 45(32), pp. 16227–16238, 2020.

[77] M. L. Sabo, N. Mariun, H. Hizam, M. A. MohdRadzi, A. Zakaria, "Spatial matching of large-scale grid-connected photovoltaic power generation with utility demand in Peninsular Malaysia," *Applied Energy*, vol. 191, pp. 663–688, 2017.

[78] M. Mousavi, H. Gitinavard, S. Mousavi, "A soft computing based-modified ELECTRE model for renewable energy policy selection with unknown information," *Renewable and Sustainable Energy Reviews*, vol. 68, pp. 774–787, 2017.

[79] M. Sabitha, M. Mayilvahanan, "Application of Dimensionality Reduction techniques in Real time Dataset," *International Journal of Advanced Research in Computer Engineering & Technology*, vol. 5(7), pp. 2187–2189, 2016.

[80] M. Uyan, "GIS-based solar farms site selection using analytic hierarchy process (AHP) in Karapinar region, Konya/Turkey," *Renewable and Sustainable Energy Reviews*, vol. 28, pp. 11–17, 2015.

[81] M. W. Tian, H. C. Yuen, S. R. Yan, W. L. Huang, "The multiple selections of fostering applications of hydrogen energy by integrating economic and industrial evaluation of different regions", *International Journal of Hydrogen Energy*, vol. 44(56), pp. 29390–29398, 2019.

[82] N. C. Onat, S. Gumus, M. Kucukvar, O. Tatari, "Application of the TOPSIS and intuitionistic fuzzy set approaches for ranking the life cycle sustainability performance of alternative vehicle technologies," *Sustainable Production and Consumption*, vol. 6, pp. 12–25, 2016.

[83] N. Gandotra, R. K. Bajaj, J. Mathew, "On ranking in triangular intuitionistic fuzzy multi criteria decision making under (α, β) cut with useful parametric entropy," *Proceeding of the International Conference on Advances in Computing and Communications*, pp. 69–74, 2016.

[84] N. K. Pal, S. K. Pal, "Object background segmentation using new definition of entropy," *IEEE Proceeding*, 136(E), pp. 284–295, 1989.

[85] N. M. A Huijts, B. Van Wee, "The evaluation of hydrogen fuel stations by citizens: the interrelated effects of sociodemographic, spatial and psychological variables," *International Journal of Hydrogen Energy*, vol. 40, pp. 10367–10381, 2015.

[86] O. Parkash, P. Sharma, R. Mahajan, "New measures of weighted fuzzy entropy and their applications for the study of maximum weighted fuzzy entropy principle," *Information Sciences*, vol. 178(11), pp. 2389–2395, 2008.

[87] P. A. Pilavachi, S. D. Stephanidis, V. A. Pappas, N. H. Afgan, "Multi-criteria evaluation of hydrogen and natural gas fuelled power plant technologies," *Applied Thermal Engineering*, vol. 29(11-12), pp. 2228–2234, 2009.

[88] P. Angolucci, "Hydrogen infrastructure for the transport sector," *International Journal of Hydrogen Energy*, vol. 32, pp. 3526–3544, 2007.

[89] P. Bellany, P. Upham, R. Flynn, M. Ricci, "Unfamiliar fuel: how the UK public view the infrastructure required to supply hydrogen for road transport," *International Journal of Hydrogen Energy*, vol. 41, pp. 6534–6543, 2016.

[90] P. Chatterjee, S. Mondal, S. Boral, A. Banerjee, S. Chakraborty, "A novel hybrid method for non-traditional machining process selection using factor relationship and Multi-Attributive Border Approximation Method," *Facta Universitatis, series: Mechanical Engineering*, vol. 15, pp. 439–456, 2017.

[91] P. Hennicke, M. Fischedick, "Towards sustainable energy systems: The related role of hydrogen," *Energy Policy*, vol. 34, pp. 1260–1270, 2006.

[92] P. K. Maji, R. Biswas, A. R. Roy, "Intuitionistic fuzzy soft sets," *Journal of fuzzy mathematics*, vol. 9, pp. 677–692, 2001.

[93] P. K. Maji, R. Biswas, A. R. Roy, "An application of soft sets in a decision making problem," *Computers and Mathematics with Applications*, vol. 44, pp. 1077–1083, 2002.

[94] P. K. Maji, R. Biswas, A. R. Roy, "Soft Set Theory," *Computers and Mathematics with Applications*, vol. 45, pp. 555–562, 2003.

[95] P. K. Bhatia, Singh S., "Three families of generalized fuzzy directed divergence," *Advanced Modelling and Optimization*, vol. 14(3), pp. 599–614, 2012.

[96] P. Rani, A. R. Mishra, K. R. Pardasani, A. Mardani, H. Liao, D. Streimikiene, "A novel VIKOR approach based on entropy and divergence measures of Pythagorean fuzzy sets to evaluate renewable energy technologies in India," *Journal of Cleaner Production*, vol. 238, 2019, [DOI:https://doi.org/10.1016/j.jclepro.2019.117936].

[97] P. Ren, Z. Xu, X. Gou, "Pythagorean fuzzy TODIM approach to multi-criteria decision making," *Applied Soft Computing*, vol. 42, pp. 246–259, 2016.

[98] Q. Zhang, S. Jiang, "A note on information entropy measure for vague sets," *Information Sciences*, vol. 178, pp. 4184–4191, 2008.

[99] R. Benayoun, B. Roy, B. Sussman, "ELECTRE: Une methode pour guider le choix en presence de points de vue multiples," *Note de travail 49, Direction Scientifique: SEMA-METRA International*, 1966.

[100] R. Joshi, S. Kumar, D. Gupta, H. Kaur, "A Jensen α-norm Dissimilarity Measure for Intuitionistic Fuzzy Sets and Its Applications in Multiple Attributes Decision Making," *International Journal of Fuzzy Systems*, vol. 20(4), pp. 1188–1202, 2018.

[101] R. Joshi, S. Kumar, "An (R', S')-norm fuzzy relative information measure and its applications in strategic decision making," *Computational and Applied Mathematics*, vol. 37, pp. 4518–4543, 2018.

[102] R. Joshi, S. Kumar, "(R, S)-norm information measure and a relation between coding and questionaire theory," *Open systems and Information Dynamics*, vol. 23(3), pp. 1–12, 2016.

[103] R. H. Lin, Z. Z. Ye, B. D. Wu, "A review of hydrogen station location models", *International Journal of Hydrogen Energy*, 2020, https://doi.org/10.1016/j.ijhydene.2019.12.035.

[104] R. K. Bajaj, T. Kumar, N. Gupta, "R-norm intuitionistic fuzzy information measures and its computational application," *Eco-friendly Computing and Communication Systems, Communications in Computer and Information Science*, vol. 305, pp. 372–380, 2012.

[105] R. R. Yager and A. M. Abbasov, "Pythagorean membership grades, complex numbers and decision making," *International Journal of Intelligent Systems*, vol. 28, pp. 436–452, 2014.

[106] R. R Yager, "Pythagorean fuzzy subsets," in *Proceedings of Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton*, Canada, pp. 57–61, 2013.

[107] R. R. Yager, "Pythagorean Membership Grades in Multicriteria Decision Making," *IEEE Transactions on Fuzzy Systems*, vol. 22(4), pp. 958–965,2014.

[108] R. Verma, B. D. Sharma, "Intuitionistic fuzzy Jensen Renyi divergence, application to multiple attribute decision making," *Informatica*, vol. 37(4), pp. 399–409, 2013.

[109] R. Verma, B. D. Sharma, "On generalized intuitionistic fuzzy divergence (relative information) and their properties," *Journal of Uncertain Systems*, vol. 6(4), pp. 308–320, 2012.

[110] R. Verma, B. D. Sharma, "Exponential entropy on intuitionistic fuzzy sets," Kybernetika, vol. 49(1), pp. 114–127, 2013.

[111] R. Zimmer, J. Welke, "Let's go green with hydrogen! the general public's perspective," *International Journal of Hydrogen Energy*, vol. 37, pp. 17502–17508, 2012.

[112] S. Hardman, E. Shiu, R. Steinberger-Wilckens, T. Turrentine, "Barriers to the adoption of fuel cell vehicles: a qualitative investigation into early adopters attitudes," *Transportation Research Part A: Policy and Practice*, vol. 95, pp. 166–182, 2017.

[113] S. F. Zhang, S. Y. Liu, "GRA-based intuitionistic multi criteria decision making method for personal selection," *Expert Systems with Application*s, vol. 38(9), pp. 11401–11405, 2011.

[114] S. Kullback, R. A. Leibler, "On information and sufficiency", *The Annals of Mathematical Statistics*, vol. 22, pp. 79–86, 1951.

[115] S. K. De, R. Biswas, A. R. Roy, "An application of intuitionistic fuzzy sets in medical diagnosis," *Fuzzy Sets and Systems*, vol. 117(2), pp. 209–213, 2001.

[116] S. Montes, I. Couso, P. Gil, C. Bertoluzza, "Divergence measure between fuzzy sets," *International Journal of Approximate Reasoning*, vol. 30(2), pp. 91–105, 2002.

[117] S. Narayanamoorthy, L. Ramya, D. Baleanu, J. V. Kureethara, V. Annapoorani, "Application of normal wiggly dual hesitant fuzzy sets to site selection for hydrogen underground storage", *International Journal of Hydrogen Energy*, vol. 44(54), pp. 28874–28892, 2019.

[118] S. Opricovic, "Multicriteria optimization of civil engineering systems," *Faculty of Civil Engineering, Belgrade*, vol. 2(1), pp. 5–21, 1998.

[119] S. Opricovic, G. H. Tzeng, "Extended VIKOR method in comparison with outranking methods," *European Journal of Operational Research*, vol. 178, pp. 514–529, 2007.

[120] S. Ozdemir, G. Sahin, "Multi-criteria decision-making in the location selection for a solar PV power plant using AHP," *Measurement*, vol. 129, pp. 218–226, 2018.

[121] S. Zeng, J. Chen, X. Li, "A hybrid method for Pythagorean fuzzy multiple criteria decision making," *International Journal of Information Technology & Decision Making*, vol. 15(2), pp. 403–422, 2016.

[122] T. Chen, C. Li, "Determining objective weights with intutionistic fuzzy entropy measures: A comparative analysis," *Information Sciences*, vol. 180, pp. 4207–4222, 2010.

[123] T. Demirel, U. Yalcinn, "Multi-Criteria Wind Power Plant Location Selection using Fuzzy AHP," in *Proceedings of the 8th International FLINS Conference*, Madrid, Spain, 21–24 September, 2008.

[124] T. Kaya, C. Kahraman, "Multicriteria decision making in energy planning using a modified fuzzy TOPSIS methodology," *Expert Systems with Applications*, vol. 38(6), pp. 6577–6585, 2011.

[125] T. L. Saaty, "The analytical hierarchy process," McGraw-Hill: New York, NY, USA, 1980.

[126] The Electrical Portal, Site selection of Hydroelectric power plant. Available online: *http://www.theelectricalportal.com/*2015/08/*site−selection−of−hydroelectric−power.html*.

[127] V. Eveloy, T. Gebreegziabher, "A Review of Projected Power-to-Gas Deployment Scenarios," *Energies*, vol. 11, 1824, 2018.

[128] V. Q. Gokalp, E. Lebas, "Alternative fuels for industrial gas turbines (AFTUR)," *Applied Thermal Engineering*, vol. 24, pp. 1655–1663, 2004.

[129] W. L. Hung, M. S. Yang, "Fuzzy entropy on intuitionistic fuzzy sets," *International Journal of Intelligent Systems*, vol. 21 pp. 443–451, 2006.

[130] W. K. M. Brauers, E. K. Zavadskas, "The MOORA method and its application to privatization in transition economy," *Control and Cybernetics*, vol. 35(2), pp. 443–468, 2006.

[131] W. L. Hung, M. S. Yang, "On the j-divergence of intuitionistic fuzzy sets and its application to pattern recognition", *Information Science*, vol. 178(6), pp. 1641–1650, 2008.

[132] W. L. Hung, M. S. Yang, "Similarity measures of intuitionistic fuzzy sets based on Housedorff distance," *Pattern Recognition Letters*, vol. 25, pp. 1603–1611, 2004.

[133] W. R. W. Mohd, L. Abdullah, "Similarity measures of Pythagorean fuzzy sets based on combination of cosine similarity measure and Euclidean distance measure," *AIP Conference Proceedings*, 1974, 030017, 2018.

[134] W. Pedrycz, "Fuzzy sets in pattern recognition, accomplishments and challenges," *Fuzzy Sets and System*, vol. 90(2), pp. 171–176, 1997.

[135] W. Q. Wang, X. L. Xin, "Distance measures between intuitionistic fuzzy sets," *Pattern Recognition Letters*, vol. 26, pp. 2063–2069, 2005.

[136] X. D. Peng, S. Ganeshsree, "Pythagorean fuzzy set: state of the art and future directions," *Artificial Intelligence Review*,vol. 52, pp.1873–1927, 2019.

[137] X. G. Shang, W. S. Jiang, "A note on fuzzy information measure," *Pattern Recognition Letters*, vol. 18, pp. 425–432, 1997.

[138] X. L. Zhang, "A Novel Approach Based on Similarity Measure for Pythagorean Fuzzy Multiple Criteria Group Decision Making," *International Journal of Intelligent Systems*, vol. 31(6), pp. 593–611, 2016.

[139] X. L. Zhang, Z. S. Xu, "Extension of TOPSIS to multiple-criteria decision making with Pythagorean fuzzy sets," *International Journal of Intelligent Systems*, vol. 29, pp. 1061–1078, 2014.

[140] X. Peng, Y. Yang, J. Song, Y. Jiang, "Pythagorean Fuzzy Soft Set and Its Application," *Computer Engineering*, vol. 41, pp. 224–229, 2015.

[141] X. Peng, H. Yuan, Y. Yang, "Pythagorean Fuzzy Information Measures and their applications," *International Journal of Intelligent Systems*, vol. 32(10), pp. 991–1029, 2017.

[142] X. Peng, Y. Yang, "Some results for Pythagorean fuzzy sets," *International Journal of Intelligent Systems*, vol. 30, pp. 1133–1160, 2015.

[143] X. Xu, T. Liang, J. Zhu, D. Zheng, T. Sun, "Review of Classical Dimensionality Reduction and Sample Selection Methods for Large-scale Data Processing," *Neurocomputing*, vol. 328, pp. 5–15, 2019.

[144] Y. C. Jiang, Y. Tang, J. Wang, S. Tang, "Reasoning with intuitionistic fuzzy rough description logics," *Information Sciences*, vol. 179, pp. 2362–2378, 2009.

[145] Y. Noorollahi, H. Yousefi, M. Mohammadi, "Multi-criteria decision support system for wind farm site selection using GIS," *Sustainable Energy Technologies and Assessments*, vol. 13, pp. 35–50, 2016.

[146] Y. Yong, J. Chenli, "Fuzzy Soft Matrices and their Applications," *Lecture notes in computer Science*, vol. 7002, pp. 618–627, 2011.

[147] Z. M. Ma, Z. S. Xu, "Symmetric Pythagorean Fuzzy Weighted Geometric/Averaging Operators and Their Application in Multicriteria Decision-Making Problems," *International Journal of Intelligent Systems*, vol. 31(12), pp. 1198–1219, 2016.

[148] Z. P. Fan, "Complicated multiple attribute decision making: theory and applications", Ph.D. Dissertation, Northeastern university, Shenyang, China, 1996.