SYMMETRY ANALYSIS OF SOME FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS

Thesis submitted in fulfillment of the requirements for the Degree of

DOCTOR OF PHILOSOPHY

BY

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DECLARATION BY THE SCHOLAR

I hereby declare that the work reported in the Ph.D. thesis entitled, **"SYMMETRY ANALYSIS OF SOME FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS"** submitted at **Jaypee University of Information Technology, Waknaghat, India**, is an authentic record of my work carried out under the supervision of **Prof. Karanjeet Singh**. I have not submitted this work elsewhere for any other degree or diploma.

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CERTIFICATE

This is to certify that the thesis entitled, **"SYMMETRY ANALYSIS OF SOME FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS"** which is being submitted by **Manoj Gaur** in fulfillment for the award of degree of **Doctor of Philosophy** in **Mathematics** by the **Jaypee University of Information Technology, Waknaghat, India** is the record of candidate's own work carried out under my supervision. This work has not been submitted partially or wholly to any other University or Institute for the award of this or any other degree or diploma.

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(Manoj Gaur)

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Abstract

The objective of the thesis entitled "Symmetry Analysis of Some Fractional Order Partial Differential Equations", is to study the applications of Lie group theory to the nonlinear fractional order partial differential equations (FPDEs) or their systems which represent some of the important physical phenomena. Our primary objective in this thesis is to identify the symmetries of FPDEs in order to obtain exact solutions, which are useful in studying the integrability and physical behaviour of the equations.

During the past few decades fractional calculus has grown predominantly in pure mathematics as well as in scientific applications, due to the fact that many processes in physics and engineering can be modeled more accurately by fractional derivatives or fractional integrals than the traditional integer order derivatives or integrals. In many applications, it is presumed that the future state of a system is independent of the past state and determined entirely on the present. But now it has been recognised that this assumption leads to first approximation of the true situations. Therefore, for a better approximation one has to consider also the past history of the system. This can be achieved by using the fractional order differential operator which is not local in nature, i.e., the derivative depends on the whole history of the function. The exact solutions of these fractional differential equations play a central role in the theories of these physical phenomena and have become more and more sought after during last few decades. Lie group method is one of the mathematical techniques which is applicable to all types of differential equations to furnish a variety of exact solutions in a systematic manner. The investigations carried out in this thesis are confined to the applications of Lie group methods to the five nonlinear FPDEs viz. space-time fractional Burgers-Poisson (FBP) equation, time fractional potential Burgers' equation, variable coefficient space-time fractional potential Burgers' (FPB) equation, time fractional Gardner and space-time fractional coupled KdV equation.

Chapter 2 deals with the study of following fractional order Burgers-Poisson

equation(FBP equation)

$$u_t^{(\alpha)} - (u_x^{(2\beta)})_t^{\alpha} + u_x^{(\beta)} + uu_x^{(\beta)} - 3(u_x^{(\beta)}u_x^{(2\beta)} + uu_x^{(3\beta)}) = 0,$$

where $x \in (0, \infty),\ \ t > 0,\ \ 0 < \alpha, \beta < 1.$

On carrying over the Lie group method to FBP equation, the groups of transformations admitted by the equation under consideration have been derived. Consequently, by using the symmetries involving arbitrary parameter, the FBP equation has been reduced to ODE which is again studied for group invariant solutions.

Chapter 3 is devoted to the study of following time fractional Potential Burgers' equation:

$$u_t^{(\alpha)} = Au_{xx} + B(u_x)^2, \quad x \in (0, \infty),\ t > 0,\ 0 < \alpha < 1,$$

where A and B are real constant parameters.

The time fractional Potential Burgers' equation is reduced to an ordinary differential equation of fractional order corresponding to the Erdélyi-Kober fractional derivative by using Lie classical symmetries. Further, an analytic solution is furnished by means of the Invariant Subspace Method.

In Chapter 4, a study has been made on the following space-time fractional Potential Burgers' equation for invariance under continuous group of transformations via Lie classical approach:

$$u_t^{(\alpha)} = f(t)u_x^{(2\beta)} + g(t)\left(u_x^{(\beta)}\right)^2, \quad x \in (0, \infty),\ t > 0,\ 0 < \alpha, \beta < 1,$$

where $f(t)$ and $g(t)$ are arbitrary functions of t.

For the FPB equation, six-dimensional symmetries have been obtained and using the subalgebras of Lie algebras, it is shown that there are various group theoretic reductions of this equation depending on certain choices of infinitesimal generators. The reduced ODEs are investigated for group invariant solutions of FPB equation. The solutions obtained are new and involve arbitrary functions $f(t)$ and $g(t)$.

In Chapter 5, we study the classical symmetries of time fractional Gardner equation of the form:

$$u_t^{(\alpha)} = Auu_x + Bu^2u_x + u_{xxx}, \quad x \in (0,\infty),\ t > 0,\ 0 < \alpha < 1,$$

where A and B are real constant parameters. Some new solutions of time fractional Gardner equation have been reported.

Chapter 6 is devoted to the space-time fractional coupled KdV equation with time dependent coefficients

$$u_t^{(\alpha)} + f(t)uu_x^{(\beta)} + g(t)vv_x^{(\beta)} + h(t)u_x^{(3\beta)} = 0,$$

$$v_t^{(\alpha)} + \delta(t)uv_x^{(\beta)} + k(t)v_x^{(3\beta)} = 0,$$

where $x \in (0,\infty),\ \ t > 0,\ \ 0 < \alpha, \beta < 1$.

Publications Based on Present Work

1. M. Gaur, and K. Singh. "On group invariant solutions of fractional order Burgers-Poisson equation," *Applied Mathematics and Computation*, 244.1, pp. 870-877, 2014.
   [**SCIE, MathSci Net,Scopus**]

2. M. Gaur, and K. Singh. "Symmetry analysis of time fractional Potential Burgers' equation," *Mathematical Communications*, 22, pp. 1-11, 2017. (In Press)
   [**SCIE, MathSci Net,Scopus**]

3. M. Gaur, and K. Singh. "Symmetry classification and exact solutions of a variable coefficient space-time fractional Potential Burgers' equation," *International Journal of Differential Equations*, 2016.
   [**MathSci Net, Scopus**]

4. M. Gaur, and K. Singh. "Lie group of transformations of time fractional Gardner equation,"(Under Communication).

5. M. Gaur, and K. Singh. "Group classification of space-time fractional coupled KdV equation,"(Under Communication).

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Chapter 1

Introduction

1.1 Background and Motivation

The study of differential equations has been playing a central role in the development of mathematics and its applications in almost every branch of science and engineering for nearly five centuries. Most of the problems posed by nature are characteristically nonlinear and are often represented by a single or a system of partial order differential equations. Since then the integer order partial differential equations have been a powerful tool in order to model and study the dynamics of many physical processes of the applied sciences. But nature often presents complex dynamics, which cannot be explained by means of ordinary models and from the experimental observations and reality, it has been revealed that there exists a lot of complex systems in nature which have anomalous dynamics such as the transport of chemical contaminants, the dynamics of viscoelastic materials as polymers, network traffic, financial markets and many more. In most of the above mentioned cases, their dynamics cannot be characterized by classical derivative models. During the last four decades the fractional derivatives have been proved to be valuable tools in the modelling of such physical phenomena. Whilst extensive studies have been made for the integer derivative models, on the other hand it remains much harder to understand fractional order models because of their inherent complexity and the lack of their simple superposed solutions. The field of fractional differential equations (FDEs) is very wide-ranging and needs to be explored in great detail due to the fact that several complex physical phenomena can be effectively represented by these equations. It is clear that the attempts to understand the nonlinear world using fractional models will dominate a large part of mathematical research in the years to come. The general theory and basic results for fractional differential equations have by now been thoroughly explored and are available in the form of a number of books [81, 89, 77]. The study of fractional differential equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. The construction of particular exact solutions of fractional differential equations is not an easy task and it remains a relevant problem. This is the reason why several methods to solve nonlinear fractional differential equations were recently developed in the literature, including Adomian decomposition method, fractional sub equation method, first integral method, homotopy perturbation method, Lie group theory method and so on (see for example [86, 20, 91, 101, 74, 79, 99, 13]). Most recently, according to invariance principles, the invariant subspace method established by V.A. Galaktionov and S.R. Svirshchevski [34] to study partial differential equations was generalised by R.K. Gazizov and A.A. Kasatkin [41] to construct some exact solutions for time fractional differential equations. The fractional versions of the well-known equations of applied mathematics, such as the growth equation, diffusion equation, transport equation, Bloch equation, Schrödinger equation, etc., have produced many interesting solutions along with observable consequences. When compared with variety of methods available to solve a system of integer order partial differential equations, the tools for analysis of fractional order partial differential equations (FPDEs) are limited to some very special categories. In a sense fractional order systems must be treated in toto and in their full complexity, and so it is not surprising that there exists no general method for solving them.

The study of exact solutions of fractional differential equations has not only provided information about the phenomena but has, in fact, helped in making more precise some of the concepts and theories developed in last few decades. These solutions provide more information about the phenomena in various aspects and often with several important physical parameters, prove useful to discuss and examine the sensitivity of physical phenomena they describe. The exact solutions are also supportive in designing and testing of numerical algorithms. Exact solutions for fractional order differential equations are rare, and the methods, which can generate families of them, are not only increasingly popular, but more and more sought. Lie group method is one of the mathematical techniques which is applicable to all types of differential equations to furnish a variety of exact solutions in a systematic manner [58, 84]. A number of excellent texts and survey articles have concentrated on the discussions of symmetry analysis for integer order differential equations. On the contrary, symmetry analysis studies of FDEs are pretty new. Up to now, in literature, solely (1+1) dimensional evolution type equations have been studied. Some of these studies have been made employing the modified Riemann-Liouville operator where explicit and exact solutions are obtained. In other studies utilizing the Riemann-Liouville operator, FPDEs are reduced to FDEs with Erdélyi-Kober fractional differential operator. Some of the properties of the fractional derivatives are very different from the classical ones; therefore, there exists a huge motivation to dig into area of finding the symmetries of some fractional differential equations.

In 2007, Gazizov generalised the method of Lie groups to investigate the continuous transformation groups of fractional differential equations and proposed some prolongation formulae [38]. Further in 2010, Wu [103] introduced a fractional Lie group method for anomalous diffusion equations and evaluated some non–differentiable solutions based on modified Riemann–Liouville derivative [62, 63, 64].

The work carried out in this thesis is dedicated to the applications of group theoretic techniques based on the theory of continuous group of point transformations also known as Lie groups acting on the space of independent and dependent variables of the system. The method was introduced originally by Sophus Lie [72].

Lie established that the order of an ordinary differential equation can be reduced by one if it is invariant under a one parameter symmetry group and for a partial differential equation the invariance under a continuous group of transformations leads directly to superposition of solutions in terms of transformations [73]. Further, Ovsiannikov [85], Bluman & Cole [13] and Olver [84] extended the theory of Lie groups to wide range of problems.

The prime objective and motivation behind the proposed study is to demonstrate the importance and efficacy of symmetry group methods in solving fractional systems. In brief, a symmetry group of a single or a system of partial differential equation of fractional order is a continuous group of transformations acting on the space of independent and dependent variables which leaves the equation(s) invariant. This group can be determined algorithmically and then the solutions of fractional order partial differential equation(s) can be found by solving a reduced system of ordinary differential equations of fractional order. The theory and applications of Lie groups may be obtained in excellent text such as those of Bluman and Cole [9], Olver [82], Ovsianikov[85] and Ibragimov [58].

1.2 Methodology

The Lie group method of differential equations was originally established and applied by Sophus Lie [72, 73] during the period 1872-1899. Regardless of its important features, the Lie's approach to differential equations faded in to obscurity and the entire subject lay dormant for almost half a century. It was in the fifties of last century, when the work of G. Birkhoff [8] and I. Sedov [92] on dimensional analysis gave relevant attention to the unexploited applications of Lie groups to the differential equations and then, it was successfully applied to wide range of problems through the pioneering efforts of Ovsiannikov [85] and his co-workers in the late 1950s. By the late 1960s and early 1970s, the whole field was active again and new applications of group theory were being developed by a number of researchers including Bluman and Cole [9, 13], Bluman and Anco

[11], Bluman and Kumei [10], Cantwell [19], Stephani [94], Hydon [57], Olver and his co-workers [82, 83, 84], Ibragimov [58, 59], Ibragimov and Kovalev [60], Bhutani et al. [14, 15, 16], Grundy [46], Hill et al. [52, 53, 54, 55], Clarkson and Mansfield [23, 24], Gagnon and Winternitz [32]. Lie group method of differential equations provides an essential framework to examine in a systematic way a wide range of topics such as the integration by quadrature of ordinary differential equations, homogeneous and separable equations, methods of undetermined coefficients, reduction of order, the determination of invariant solutions of initial and boundary value problems, derivation of conservation laws, construction of links between different differential equations that turn out to be equivalent. Lie has established that the invariance of an ODE underone-parameter group of transformations, provides some special solutions called invariant solutions without knowledge of the general solution of the ODE. For an exhaustive review of Lie's work on this aspect, we refer to the works of Lie and Engel [73], Cohen [25], Goursat [45], Dickson [27], Ince [61] and Heremann and Heremann [51]. The key idea of Lie's theory of symmetry analysis of differential equations relies on the invariance of the latter under a transformation of independent and dependent variables. This transformation forms a local group of point transformations establishing a diffeomorphism on the space of independent and dependent variables, mapping solutions of the equations to other solutions. Any transformation of the independent and dependent variables in turn induces a transformation of the derivatives. Lie showed that the problem of finding the group of point transformations leaving invariant a differential equation (ordinary or partial), i.e., a point symmetry of a differential equation (DE), reduced to solving related linear systems of determining equations for its infinitesimal generators. He also showed that a point symmetry of a DE leads, in the case of an ordinary differential equation, to reducing the order of the DE and in the case of a partial differential equation, to finding special solutions called invariant (similarity) solutions of the DE. In this direction, some other important and significant contributions are from Gandarias and Bruzon [36, 37], Rosati and Nucci [88], Bihlo and Popvych [6, 7], Anco and Dennis [1]. General theories of infinite-dimensional Lie groups

and algebras [65], arising in relativity, field theory, fluid mechanics, solitons, and geometry, remain knotty. Higher order or generalized symmetries, in which the infinitesimal generators also depend upon derivative coordinates, first proposed by Noether [80] have been used to classify integrable (soliton) systems. Recursion operators are used to generate such higher order symmetries, and, via Noether's theorem, higher order conservation laws [84]. Among various generalizations of Lie's classical theory there are the following techniques:

1. Nonclassical method [9]
2. General method of differential constraints [87, 82]
3. Introduction of approximate symmetries [60, 3]
4. Generalized symmetries [84]
5. Equivalence transformations [75]
6. Nonlocal symmetries [10, 71, 84]

In recent years, Lie's classical theory has gained much interest of many researchers in the field of fractional differential equations (FDEs). The prime motivation in carrying out this study has been to demonstrate the importance and efficacy of the Lie group method over various other methods available in literature. Some specific physical sytems, governed by nonlinear fractional order partial differential equations have been considered to accomplish the task. The description of the various systems studied and forming the subject of investigation for different chapters is made in brief in section (1.8). The problems studied are dealt with in two phases - in the first, the symmetries of the sytem under investigation are derived using the Lie group method and then in the second phase, after successful deduction of the reduced systems of ODEs, the efforts are confined to furnish the exact solutions. In some problems, we have also investigated some other exact solutions using the invariant subspace method. After giving a brief survey of the available literature relevant to the work put up in chapters, we reproduce in the following sections, certain characterstic features of the techniques utilized, general notions essential for understanding and carry over of the Lie group method and Invariant subspace method to furnish exact solutions of FPDEs.

1.3 Lie Group Method to Construct Solutions of FPDEs

In the thesis, we deal with the method of group invariant solutions, based on the theory of continuous group of point transformations also known as Lie groups acting on the space of independent and dependent variables of the system. The method was introduced originally by Sophus Lie [72, 73]. Lie established that the order of an ordinary differential equation can be reduced by one if it is invariant under a one parameter symmetry group and for a partial differential equation the invariance under a continuous group of transformations leads directly to superposition of solutions in terms of transformations. In the following sections, we will firstly introduce the relevant concepts of the Lie group of transformations and then we will provide an algorithmic description of the techniques which are applied in the later chapters to derive the symmetry group of the systems under consideration. The method of invariant subspace, which is of interest in the present work for obtaining some other exact solutions, is also presented in a concise manner emphasizing the application procedure. For details on Lie group method, various theorems, their proofs and other concepts, we refer our reader to Olver [84], Bluman and Cole [13]. Also the corresponding details on the fractional differential equations and the invariant subspace method can be found in [81, 86, 89, 34, 38, 90]. We first present some fundamentals of Lie group theory (refer to sections (1.3.1) to (1.5)).

1.3.1 Lie Group of Transformations

Definition 1.3.1 *Let* $\mathbf{x} = (x_1, x_2, ..., x_n)$ *lie in a region* $D \subset \mathbb{R}^n$. *Consider a one parameter family of transformations*

$$\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x}; \epsilon), \tag{1.3.1}$$

defined for each $\mathbf{x}$ in D and parameter $\epsilon \in G \subset \mathbb{R}$, with $\phi(\epsilon, \delta)$ defining a law of composition of parameters ϵ and δ in G, such that

1. *For each ϵ in G, the transformations are bijective on D.*

2. *G with the law of composition ϕ forms a group.*

3. *For each $\mathbf{x}$ in D, $\tilde{\mathbf{x}} = \mathbf{x}$ when $\epsilon = \epsilon_0$ corresponds to the identity element of G, i.e.,*

$$\mathbf{X}(\mathbf{x}; \epsilon_0) = \mathbf{x}.$$

4. *If $\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x}; \epsilon)$, $\tilde{\tilde{\mathbf{x}}} = \mathbf{X}(\tilde{\mathbf{x}}; \delta)$, then*

$$\tilde{\tilde{\mathbf{x}}} = \mathbf{X}(\mathbf{x}; \phi(\epsilon, \delta)).$$

Such family of transformations is called a one-parameter group of transformations.

Definition 1.3.2 *A one-parameter group of transformations (1.3.1) defines a one-parameter Lie group of transformations if, in addition to satisfying axioms (1)–(4) of definition 1.3.1., the followings hold:*

1. *ϵ is a continuous parameter, i.e., G, is an interval in $\mathbb{R}$. Without loss of generality, $\epsilon = 0$ corresponds to the identity element ϵ_0.*

2. *$\mathbf{X}$ is infinitely differentiable function of $\mathbf{x}$ in D and an analytic function of ϵ in G.*

3. *$\phi(\epsilon, \delta))$ is an analytic function of ϵ and δ in G.*

1.3.1.1 Infinitesimal Form of a Lie Group

Expanding $\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x};\epsilon)$ about $\epsilon = 0$, we get

$$\begin{aligned}\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x};\epsilon) = \mathbf{X}(\mathbf{x};0) + \epsilon\frac{\partial \mathbf{X}}{\partial \epsilon}|_{\epsilon=0} + \frac{1}{2}\epsilon^2\frac{\partial^2 \mathbf{X}}{\partial \epsilon^2}|_{\epsilon=0} + ...\\ = \mathbf{x} + \epsilon\frac{\partial \mathbf{X}}{\partial \epsilon}|_{\epsilon=0} + o(\epsilon^2).\end{aligned}$$

The transformation $\tilde{\mathbf{x}} = \mathbf{x} + \bar{\xi}(\mathbf{x})\epsilon$ defines the infinitesimal transformation of Lie group of transformations (1.3.1) and the components of $\bar{\xi}(\mathbf{x})$ are called the infinitesimals, where

$$\bar{\xi}(\mathbf{x}) = \frac{\partial \mathbf{X}}{\partial \epsilon}|_{\epsilon=0}.$$

1.3.1.2 Infinitesimal Generators

The linear differential operator

$$V = \bar{\xi}(\mathbf{x}).\nabla = \xi_i(\mathbf{x})\frac{\partial}{\partial x_i}$$

with summation over a repeated index, is called the infinitesimal generator of the Lie group of transformations (1.3.1). Here, ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right).$$

1.3.1.3 Invariant Functions

An infinitely differentiable function $F(\mathbf{x})$ is called an invariant of the Lie group of transformations (1.3.1) if and only if, for any group transformation (1.3.1),

$$F(\tilde{\mathbf{x}}) = F(\mathbf{x}).$$

Theorem 1.3.1 *$F(\mathbf{x})$ is invariant under a Lie group of transformations (1.3.1) if and only if, $VF(\mathbf{x}) = 0$. (For proof and more details see [84]).*

1.4 Point Transformations and Prolongations

We will be concerned with the determination of one-parameter Lie group of point transformations admitted by a given system S of fractional differential equations. A one-parameter (ϵ) Lie group of transformations is a group of transformations of the form

$$\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \mathbf{u}; \epsilon) \tag{1.4.1}$$

$$\tilde{\mathbf{u}} = \mathbf{U}(\mathbf{x}, \mathbf{u}; \epsilon), \tag{1.4.2}$$

acting on the space of $n + m$ variables

$$\mathbf{x} = (x_1, x_2, ..., x_n)$$
$$\mathbf{u} = (u^1, u^2, ..., u^m),$$

where $\mathbf{x}$ represents n independent variables and $\mathbf{u}$ denotes m dependent variables. A Lie group of point transformations (1.4.1–1.4.2) admitted by S maps any solution $\mathbf{u} = \theta(\mathbf{x})$ of S onto a one-parameter family of solutions $\mathbf{u} = \phi(\mathbf{x}; \epsilon)$ of S. Let $\partial\mathbf{u}$ denotes the set on nm coordinates corresponding to all first order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$:

$$\partial\mathbf{u} = \left(\frac{\partial u^1}{\partial x_1}, \frac{\partial u^1}{\partial x_2}, ..., \frac{\partial u^1}{\partial x_n}, \frac{\partial u^2}{\partial x_1}, \frac{\partial u^2}{\partial x_2}, ..., \frac{\partial u^2}{\partial x_n}, ..., \frac{\partial u^m}{\partial x_1}, \frac{\partial u^m}{\partial x_2}, ..., \frac{\partial u^m}{\partial x_n}\right). \tag{1.4.3}$$

In general, for $k \geq 1$, let $\partial^k\mathbf{u}$ denote the set of coordinates

$$u^\mu_{i_1,i_2,...,i_k} = \frac{\partial^k u^\mu}{\partial x_{i_1} \partial x_{i_2} ... \partial x_{i_k}},$$

with $\mu = 1, 2, ..., m$ and $i_j = 1, 2, ..., n$ for $j = 1, 2, ..., k$ corresponding to all kth-order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$.

It turns out that the natural transformation of partial derivatives of the dependent variables leads successively to extensions (prolongations) of a one-parameter Lie group of transformations (1.4.1–1.4.2) acting on $(\mathbf{x}, \mathbf{u})$-space to one-parameter Lie groups of transformations acting on $(\mathbf{x}, \mathbf{u}, \partial\mathbf{u})$-space, $(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u})$-space,..., $(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ...\partial^k\mathbf{u})$-space for any $k > 2$. [For a given system S of differential equations, k would be the order of the highest order derivative appearing in S]. Then the infinitesimal transformations of (1.4.1–1.4.2) is naturally extended successively to infinitesimal transformations acting on $(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ...\partial^l\mathbf{u})$-space, $l = 1, 2, ..., k$.

1.4.1 Extended Infinitesimal Transformations

In the study of system of PDEs, the situation of m dependent variables $\mathbf{u} = (u^1, u^2, ..., u^m)$ and n independent variables $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{u} = \mathbf{u}(\mathbf{x})$, with $m \geq 2$, arises. This leads to consideration of extended transformations from $(\mathbf{x}, \mathbf{u})$-space to $(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ...\partial^k\mathbf{u})$-space where $\partial^k\mathbf{u}$ denotes the components of all kth-order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$. Consider the kth-extended transformation over the $(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ...\partial^k\mathbf{u})$-space

$$\tilde{x_i} = x_i + \xi_i(\mathbf{x}, \mathbf{u})\epsilon + o(\epsilon^2) \tag{1.4.4}$$

$$\tilde{u^\mu} = u^\mu + \eta^\mu(\mathbf{x}, \mathbf{u})\epsilon + o(\epsilon^2), \tag{1.4.5}$$

$$\tilde{u_i^\mu} = u_i^\mu + \eta_i^{(1)\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u})\epsilon + o(\epsilon^2), \tag{1.4.6}$$

$\vdots$

$$\tilde{u}^\mu_{i_1 i_2 \ldots i_k} = u^\mu_{i_1 i_2 \ldots i_k} + \eta^{(k)\mu}_{i_1 i_2 \ldots i_k}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u})\epsilon + o(\epsilon^2), \tag{1.4.7}$$

with the extended infinitesimals as

$$\eta_i^{(1)\mu} = D_i\eta^\mu - (D_i\xi_j)u_j^\mu$$

$$\eta^{(k)\mu}_{i_1 i_2 \ldots i_k} = D_{i_k}\eta^{(k-1)\mu}_{i_1 i_2 \ldots i_{k-1}} - (D_{i_k}\xi_j)u^\mu_{i_1, i_2, \ldots, i_{k-1}, i_k},$$

where $i_l = 1, 2, ..., n$ for $l = 1, 2, ..., k$ with $k \geq 2$ and D_i is total derivative operator defined as

$$D_i = \frac{\partial}{\partial x_i} + u_i^\mu \frac{\partial}{\partial u^\mu} + u_{ij}^\mu \frac{\partial}{\partial u_j^\mu} + u^\mu_{ii_1i_2}\frac{\partial}{\partial u^\mu_{i_1i_2}} + ... + u^\mu_{ii_1i_2\ldots i_n}\frac{\partial}{\partial u^\mu_{i_1i_2\ldots i_n}} + ...,$$

with summation over repeated index.

Here, the kth-extended infinitesimal generator is given by

$$V^{(k)} = \xi_i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i} + \eta^\mu(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu}(\mathbf{x},\mathbf{u},\partial\mathbf{u})\frac{\partial}{\partial u_i^\mu} +$$
$$\eta_{i_1 i_2}^{(2)\mu}(\mathbf{x},\mathbf{u},\partial\mathbf{u},\partial^2\mathbf{u})\frac{\partial}{\partial u_{i_1 i_2}^\mu} + ... + \eta_{i_1 i_2 ... i_k}^{(k)\mu}(\mathbf{x},\mathbf{u},\partial\mathbf{u},\partial^2\mathbf{u},...,\partial^k\mathbf{u})\frac{\partial}{\partial u_{i_1 i_2 ... i_k}^\mu}, \quad k \geq 1.$$

1.4.1.1 The Invariance Condition for a System of PDEs

Lie symmetry of a differential equation is a one parameter point transformation which leaves the differential equation invariant. Consider a system of N PDEs with n independent variables $\mathbf{x} = (x_1, x_2, ..., x_n)$ and m dependent variables $\mathbf{u} = (u^1, u^2, ..., u^m)$, given by

$$F^\mu(\mathbf{x},\mathbf{u},\partial\mathbf{u},\partial^2\mathbf{u},...,\partial^k\mathbf{u}) = 0, \quad \mu = 1, 2, ..., N. \tag{1.4.8}$$

Definition 1.4.1 *A one-parameter Lie group of point transformations (1.4.4)–(1.4.5) leaves the system of PDEs (1.4.8) invariant iff its kth extension, defined by (1.4.4)–(1.4.7), leaves invariant the N surfaces in $(\mathbf{x},\mathbf{u},\partial\mathbf{u},\partial^2\mathbf{u},...,\partial^k\mathbf{u})$-space, defined by (1.4.8).*

Theorem 1.4.1 *(Infinitesimal Criterion for the Invariance of a System of PDEs). Let*

$$V = \xi_i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i} + \eta^\mu(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\mu}. \tag{1.4.9}$$

be the infinitesimal generator of the Lie group of point transformations (1.4.4)–(1.4.5). Let

$$V^{(k)} = \xi_i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i} + \eta^\mu(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu}(\mathbf{x},\mathbf{u},\partial\mathbf{u})\frac{\partial}{\partial u_i^\mu} +$$
$$\eta_{i_1 i_2}^{(2)\mu}(\mathbf{x},\mathbf{u},\partial\mathbf{u},\partial^2\mathbf{u})\frac{\partial}{\partial u_{i_1 i_2}^\mu} + ... + \eta_{i_1 i_2 ... i_k}^{(k)\mu}(\mathbf{x},\mathbf{u},\partial\mathbf{u},\partial^2\mathbf{u},...,\partial^k\mathbf{u})\frac{\partial}{\partial u_{i_1 i_2 ... i_k}^\mu}, \tag{1.4.10}$$

be the kth-extended infinitesimal generator of (1.4.9), with the extended infinitesimals as

$$\eta_i^{(1)\mu} = D_i\eta^\mu - (D_i\xi_j)u_j^\mu$$

$$\eta^{(k)\mu}_{i_1 i_2 \ldots i_k} = D_{i_k}\eta^{(k-1)\mu}_{i_1 i_2 \ldots i_{k-1}} - (D_{i_k}\xi_j)u^{\mu}_{i_1,i_2,\ldots,i_{k-1},i_k},$$

where $i_l = 1, 2, ..., n$ *for* $l = 1, 2, ..., k$ *with* $k \geq 2$ *and* D_i *is total derivative operator defined as*

$$D_i = \frac{\partial}{\partial x_i} + u^{\mu}_i \frac{\partial}{\partial u^{\mu}} + u^{\mu}_{ij}\frac{\partial}{\partial u^{\mu}_j} + u^{\mu}_{ii_1i_2}\frac{\partial}{\partial u^{\mu}_{i_1i_2}} + ... + u^{\mu}_{ii_1i_2\ldots i_n}\frac{\partial}{\partial u^{\mu}_{i_1i_2\ldots i_n}} + ...,$$

with summation over repeated index. Then the one-parameter Lie group of point transformations (1.4.4)–(1.4.5) is admitted by the system of PDEs (1.4.8) if and only if

$$V^{(k)}F^{\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u}) = 0, \quad \mu = 1, 2, ..., N. \tag{1.4.11}$$

when $F^{\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u}) = 0$.

1.4.1.2 Symmetry Determining Equations

Consider a system of PDEs (1.4.8) with each of its PDEs given in a solved form

$$u^{\nu_\mu}_{i_1i_2\ldots i_k} = f^{\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u}) \tag{1.4.12}$$

In terms of some specific l_μ th-order partial derivative of u^{ν_μ} for some $\nu_\mu = 1, 2, ..., m$, where $f^{\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u})$ does not depend explicitly on any of the components $u^{\nu_\sigma}_{i_1i_2\ldots i_k}$, $\sigma = 1, 2, ..., N$, for each $\mu = 1, 2, ..., N$. From theorem 1.4.1., we see that the system of PDEs (1.4.8) admits the point symmetry (1.4.9) with the kth extension given by (1.4.10), if and only if

$$\eta^{(l_\mu)\nu_\mu}_{i_1i_2\ldots i_{l_\mu}} = \xi_j\frac{\partial f^{\mu}}{\partial x_j} + \eta^{\nu}\frac{\partial f^{\mu}}{\partial u^{\nu}} + \eta^{(1)\nu}_j\frac{\partial f^{\mu}}{\partial u^{\nu}_j} + \eta^{(2)\nu}_{j_1j_2}\frac{\partial f^{\mu}}{\partial u^{\nu}_{j_1j_2}} + ... + \eta^{(k)\nu}_{j_1j_2\ldots j_k}\frac{\partial f^{\mu}}{\partial u^{\nu}_{j_1j_2\ldots j_k}}, \tag{1.4.13}$$

with

$$u^{\nu_\sigma}_{i_1i_2\ldots i_{k_\sigma}} = f^{\sigma}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u}), \quad \sigma = 1, 2, ..., N. \tag{1.4.14}$$

It is easy to see that $\eta^{(p)\nu}_{j_1j_2\ldots j_p}$ is a polynomial in the components of $\partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^p\mathbf{u}$, with coefficients that are linear homogeneous in the components of $\bar{\xi}(\mathbf{x}, \mathbf{u}), \bar{\eta}(\mathbf{x}, \mathbf{u})$ and their derivatives to order p. Thus $\bar{\xi}$ and $\bar{\eta}$ appear linearly in (1.4.13). As is the situation for a given scalar PDE, the system of symmetry determining equations (1.4.13-14) leads to a system of linear homogeneous PDEs for ξ and η.

First we eliminate the components $u^{\nu_\sigma}_{i_1 i_2 \dots i_{k_\sigma}}$ and their differential consequences from (1.4.13) by substitution from (1.4.14) and the differential consequences of (1.4.14), $\sigma = 1, 2, ..., N$. Consequently, the components of x, u and the remaining components of $\partial \mathbf{u}, \partial^2 \mathbf{u}, ..., \partial^k \mathbf{u}$ that appear in the resulting system of symmetry determining equations (1.4.13) are themselves independent variables, i.e., they take on arbitrary values. Since the resulting expression for (1.4.13) holds for any values of these independent variables, one obtains a system of linear homogeneous PDEs for ξ and η that constitutes a set of determining equations for the infinitesimal generators V admitted by the given system of PDEs (1.4.8). In particular, if each $f^\sigma(\mathbf{x}, \mathbf{u}, \partial \mathbf{u}, \partial^2 \mathbf{u}, ..., \partial^k \mathbf{u}), \quad \sigma = 1, 2, ..., N$, is a polynomial in the components of $\partial \mathbf{u}, \partial^2 \mathbf{u}, ..., \partial^k \mathbf{u}$, then the system of symmetry determining equations (1.4.13) yields polynomial equations in the independent components of $\partial \mathbf{u}, \partial^2 \mathbf{u}, ..., \partial^k \mathbf{u}$ Consequently, the coefficients of these polynomial equations must vanish separately. This yields the set of linear determining equations for $\bar{\xi}$ and $\bar{\eta}$. Typically, the numbers of determining equations are far greater than $n + m$, so that the set of determining equations is very overdetermined.

1.4.1.3 Group Invariant Solutions

Consider a system of PDEs (1.4.8) which admits a one-parameter Lie group of point transformations (1.4.4)–(1.4.5) with the infinitesimal generator (1.4.9). We assume that $\bar{\xi}(\mathbf{x}, \mathbf{u}) \neq 0$.

Definition 1.4.2 *A solution* $\mathbf{u} = \theta(\mathbf{x})$, *with components* $u^\nu = \theta^\nu(\mathbf{x})$, $\nu = 1, 2, ..., m$, *of the system of PDEs (1.4.8) is called a group invariant solution if and only if the surface* $\mathbf{u} = \theta(\mathbf{x})$ *remains invariant under the transformations (1.4.4)–(1.4.5), i.e.,*

$$\xi_i(\mathbf{x}, \theta(\mathbf{x})) \frac{\partial \theta(\mathbf{x})}{\partial x_i} = \eta^\nu(\mathbf{x}, \theta(\mathbf{x})), \ \ \nu = 1, 2, ..., m. \tag{1.4.15}$$

Equation (1.4.15) is the invariant surface condition for the invariant solutions of the system of PDEs (1.4.8) resulting from its invariance under the point symmetry

(1.4.4)–(1.4.5). As is the situation for the scalar PDE, invariant solutions can be determined by the following procedure:

1.4.1.4 Invariant Form Method

Here we first solve the invariant surface conditions (1.4.15) by explicitly solving the corresponding characteristics equations for $\mathbf{u} = \theta(\mathbf{x})$ given by

$$\frac{dx_1}{\xi_1(\mathbf{x},\mathbf{u})} = \frac{dx_2}{\xi_2(\mathbf{x},\mathbf{u})} = ... = \frac{dx_n}{\xi_n(\mathbf{x},\mathbf{u})} = \frac{du^1}{\eta^1(\mathbf{x},\mathbf{u})} = \frac{du^2}{\eta^2(\mathbf{x},\mathbf{u})} = ... = \frac{du^m}{\eta^m(\mathbf{x},\mathbf{u})}. \tag{1.4.16}$$

If $y_1(\mathbf{x},\mathbf{u}), \ . \ . \ . \ , y_{n-1}(\mathbf{x},\mathbf{u})$, $h^1(\mathbf{x},\mathbf{u})$, ..., $h^m(\mathbf{x},\mathbf{u})$, are $n+m-1$ functionally independent constants of integration that arise from solving the characteristic equations (1.4.16) with the non-zero Jacobian, i.e., $\frac{\partial(h^1,h^2,...,h^m)}{\partial(u^1,u^2,...,u^m)} \neq 0$, then the general solution $u = \theta(x)$ of the invariant surface condition equations (1.4.15) is given implicitly by the invariant form

$$u^\nu(\mathbf{x},\mathbf{u}) = \Phi^\nu(y_1(\mathbf{x},\mathbf{u}), y_2(\mathbf{x},\mathbf{u}), ..., y_{n-1}(\mathbf{x},\mathbf{u})), \tag{1.4.17}$$

where Φ^ν is an arbitrary differentiable function of its arguments, for $\nu = 1, 2, ..., m$. Note that $y_1(\mathbf{x},\mathbf{u}), \ . \ . \ . \ , y_{n-1}(\mathbf{x},\mathbf{u})$, $h^1(\mathbf{x},\mathbf{u})$, ..., $h^m(\mathbf{x},\mathbf{u})$, are $n+m-1$ functionally independent invariants of the one-parameter Lie group of point transformations with the infinitesimal generator V given by (1.4.9), and hence are $n+m-1$ canonical coordinates for the one parameter Lie group of point transformations with the infinitesimal generator V given by (1.4.9). Let $u_n(\mathbf{x},\mathbf{u})$ be the $(n+m)$th canonical coordinate satisfying $Vy_n = 1$. If the PDE system (1.4.8) is transformed by the corresponding invertible point transformation into a PDE system with independent variables $(y_1, y_2, ..., y_n)$ and dependent variables $(h^1, h^2, ..., h^m)$, then the transformed PDE system has the translation point symmetry given by

$$\tilde{y}_i = y_i, \quad i = 1, 2, ..., n-1,$$
$$\tilde{y}_n = y_n + \epsilon,$$
$$\tilde{h}^\nu = h^\nu, \quad \nu = 1, 2, ..., m.$$

Thus the variable y_n does not appear explicitly in the transformed PDE system and hence the transformed PDE system has particular solutions of the form (1.4.17) that in turn define, implicitly, specific functions $\mathbf{u} = \theta(x)$ which are invariant solutions of the PDE system (1.4.8), i.e., the PDE system (1.4.8) has invariant solutions implicitly given by the invariant form (1.4.17). In particular, these invariant solutions are found by solving a reduced system of DEs with $n-1$ independent variables $y_1, y_2, ..., y_{n-1}$ and m dependent variables $h^1, h^2, ..., h^m$. The variables $y_1, y_2, ..., y_{n-1}$ are commonly called similarity variables. The reduced system of DEs is found by substituting the invariant form (1.4.17) into the given PDE system (1.4.8). It is assumed that this substitution does not lead to a DE system with a singular equation. Note that if $\frac{\partial \xi_i}{\partial u^\mu} \equiv 0$, as is commonly the case, then $y_i = y_i(\mathbf{x}), i = 1, 2, ..., n-1$. In the case when(1.4.8) has two independent variables, i.e., $n = 2$, the reduced system of DEs is an ODE system with independent variable y_1.

1.4.1.5 Lie Algebra

For the Lie group of transformations with infinitesimal generators V_1, V_2, the commutator (Lie bracket) of V_1, V_2 is first order operator defined by

$$[V_1, V_2] = V_1V_2 - V_2V_1 \tag{1.4.18}$$

Definition 1.4.3 *A Lie algebra is a vector space $\mathbb{L}$ over $\mathbb{R}$ or $\mathbb{C}$ with a bilinear bracket operation (the commutator) satisfying the following properties:*

1. ***Bilinearity:***

$$[aV_1 + bV_2, V_3] = a[V_1, V_3] + b[V_2, V_3] \tag{1.4.19}$$

$$[V_1, aV_2 + bV_3] = a[V_1, V_2] + b[V_1, V_3] \tag{1.4.20}$$

2. ***Skew-Symmetry:***

$$[V_1, V_2] = -[V_2, V_1] \tag{1.4.21}$$

3. ***Jacobi Identity:***

$$[V_1, [V_2, V_3]] + [V_3, [V_1, V_2]] + [V_2, [V_3, V_1]] = 0. \tag{1.4.22}$$

The commutator of two vector fields again is a vector field. Moreover, if V_i and V_j are two infinitesimal generators of a symmetry transformation, the commutator of both generators will again be a generator of a symmetry group [54, 109]. As a consequence, the set of all infinitesimal generators is closed under commutation of vector fields, thus possessing more structure than just that of vector space. This additional closure property endows the space of infinitesimal generators with an additional algebraic structure, the so called Lie algebra. Hence, having found some of the infinitesimal generators V_i of an r-parameter Lie group it may be possible to find new generators by computing the commutators of the known ones. A common way to visualise the structure of a Lie algebra is the commutator table [84]. Let $V_1, V_2, ..., V_r$ be a basis of r-dimensional Lie algebra, then its commutator table has (i, j)-th entry $[V_i, V_j]$. Because the commutator is antisymmetric it suffices to compute just the part above the diagonal, as $[V_i, V_j] = -[V_j, V_i]$. The commutator table therefore reads:

Table 1.1: Commutator Table

	V_1	V_2	...	V_r
V_1	0	$[V_1, V_2]$	...	$[V_1, V_r]$
V_2	-$[V_1, V_2]$	0	...	$[V_2, V_r]$
...	...	...	...	...
V_r	-$[V_1, V_r]$	-$[V_2, V_r]$	...	0

1.5 Classical Lie Group Method: An Algorithmic Overview

The classical method essentially consists of finding symmetry reduction of PDEs with the help of determining equations obtained under the condition of invariance of the system of PDEs. More specifically, when a given system of PDES (1.4.8)is subjected to invariance under one-parameter Lie group of transformations (1.4.4)–(1.4.5), one arrives at an over determined linear system of equations for the group infinitesimals. These infinitesimals of the transformations help us obtain the reduction of the system. The stepwise procedure is as follows:
Consider a system of N PDEs with n independent variables $\mathbf{x} = (x_1, x_2, ..., x_n)$ and m dependent variables $\mathbf{u} = (u^1, u^2, ..., u^m)$, given by

$$F^\mu(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u}) = 0, \quad \mu = 1, 2, ..., N. \tag{1.5.1}$$

1. Let the one-parameter Lie group of point transformations (1.4.4)–(1.4.5) leaves the system (1.5.1) invariant

2. Apply the extended infinitesimal operator $V^{(k)}$ given by (1.4.10) to each equation of the system (1.5.1) and require that

   $$V^{(k)}F^\mu(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u})|_{F^\nu=0} = 0, \quad \mu, \nu = 1, 2, ..., N. \tag{1.5.2}$$

   The meaning of the condition (1.5.2) is that $V^{(k)}$ vanishes on the solution set of the original system (1.5.1). Precisely, this condition assures that $\mathbf{u}(\mathbf{x})$ is solution of (1.5.1) whenever $\tilde{\mathbf{u}}(\tilde{\mathbf{x}})$ is one.

3. Following the procedure as given in section (1.4.1.2), a system of linear PDEs for ξ and η that constitutes a set of determining equations for the infinitesimal generator V admitted by the system of PDEs (1.5.1) is obtained.

4. The solution of determining equations will lead to the explicit forms of ξ and η.

5. Construct the corresponding characteristic equations (1.4.16) and obtain $\mathbf{u}$ in terms of $n-1$ new independent variables.

6. Rewrite the system (1.5.1) in these new coordinates to get the reduced form of the system.

1.6 Invariant Subspace Method

One of the recently developed techniques to construct an exact solution of non-linear PDEs is the invariant subspace method and its applicability has been illustrated by many researchers, e.g. [33, 34, 76, 95, 96, 97]. This method has been extended to nonlinear FDEs [41, 48] and its applicability illustrated through the time fractional Burgers type equations. However the applicability of this method to FDEs has not been widely demonstrated. The invariant subspace method was introduced by Galaktionov [33] in order to discover exact solutions of nonlinear partial differential equations. The method was further applied by Gazizov [41] and Sahadevan [90] to some nonlinear fractional order differential equations. Here, we give a brief description of the method.
Let us consider the fractional evolution equation

$$u_t^{(\alpha)} = F[u], \tag{1.6.1}$$

where $u = u(x,t)$ is a real scalar function of two independent variables x, t and $F[u]$ is a nonlinear differential operator of order k,

$$F[u] = F(x, u_1, u_2, ..., u_k). \tag{1.6.2}$$

Here, $F(.)$ is a given sufficiently smooth function of its arguments and $u_i = \frac{\partial^i u}{\partial x^i}$, $i \geq 0$.
Let $f_1(x), ..., f_n(x)$, $N \in \mathbb{N}$ be n linearly independent functions which form an n-dimensional linear space

$$W_n = \langle f_1(x), ..., f_n(x) \rangle = \sum_{i=1}^{n} a_i f_i(x), a_i \in \mathbb{R} \tag{1.6.3}$$

that is, W_n is the linear span of $f_1(x), ..., f_n(x)$ over $\mathbb{R}$.

Definition 1.6.1 *The n-dimensional linear space* $W_n = \langle f_1(x), ..., f_n(x)\rangle$ *is called invariant under the operator* $F[u]$*, iff* $F[u] \in W_n$ *for any* $u \in W_n$*.*

Suppose that W_n is an invariant subspace with respect to a given differential F. Then there exist n functions $\phi_1, ..., \phi_n$ such that

$$F[\sum_{i=1}^{n} c_i f_i(x)] = \sum_{i=1}^{n} \phi_i(c_1, ..., c_n) f_i(x), \tag{1.6.4}$$

where $c_1, ..., c_n$ are arbitrary constants and $\{\phi_n\}$ are the expansion coefficients of $F[u] \in W_n$ in the basis $\{f_i\}$. It follows that an exact solution of fractional evolution equation (1.6.1) can be obtained as

$$u(x,t) = \sum_{i=1}^{n} a_i(t) f_i(x), \tag{1.6.5}$$

where the coefficient functions $a_1(t), a_2(t), ..., a_n(t)$ satisfy a system of fractional ODEs

$$a_i(t)_t^{(\alpha)} = \phi_i(a_1(t), a_2(t), ..., a_n(t)), \quad i = 1, 2, ..., n. \tag{1.6.6}$$

Note that an invariant subspace W_n is the space of solutions of some linear ordinary differential equation

$$L(y) = y^{(n)} + a_1(x) y^{(n-1)}, ..., a_n(x) y = 0 \tag{1.6.7}$$

for which the functions $f_i(x)$, $i = 1, 2, ..., n$, form a fundamental system of solutions. Then the invariance condition of W_n takes the form

$$L(F[y])|_{L(y)=0} = 0. \tag{1.6.8}$$

This condition leads to an over determined system for the coefficients of (1.6.7) and provides the description of all invariant spaces of given order n, [[76], [95], [96]]. The following theorem establishes the upper bound on the dimension of an invariant space.

Theorem 1.6.1 *If a linear space* W_n *is an invariant under a nonlinear differential operator* $F[y] = F(x, y, y', ..., y^{(k)})$ *of the order* k*, then* $n \leq 2k+1$*.*

For further details the reader is referred to [90].

1.7 Some Concepts from Fractional Calculus

It is worth to mention that there is no unique definition to define the fractional derivative. In the literature, different definitions for the fractional derivative such as the Caputo, the Riesz, the Grunwald-Letnikov and the Riemann- Liouville can be seen. The most popular ones are the Riemann-Liouville and the Caputo derivatives. Each fractional derivative presents some advantages and disadvantages [77, 86, 91]. The Riemann-Liouville derivative of a constant is not zero while Caputos derivative of a constant is zero but demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first order derivative might have fractional derivatives of all orders less than one in the Riemann-Liouville sense. Recently, Guy Jumarie [62, 64] proposed a simple alternative definition to the Riemann-Liouville derivative. His modified Riemann-Liouville derivative has the advantages of both the standard Riemann-Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (non-differentiable) functions and the fractional derivative of a constant is equal to zero. The work carried out in this thesis is based on some basic elements of fractional calculus, with special emphasis on the Riemann-Liouville type and modified Riemann-Liouville type derivatives [62]. We use Lie symmetries with the prolongation formula given by Gazizov et al. [38].

1.7.1 Fractional Riemann-Liouville Integral

The fractional Riemann-Liouville integral of a continuous (but not necessarily differentiable) real valued function $f(x)$ with respect to $(dx)^\alpha$ is defined as [62, 63, 77]

$$ {}_0I_x^\alpha\ f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt\ = \frac{1}{\Gamma(1+\alpha)}\int_0^x f(t)\,(dt)^\alpha\ ,\ \ 0<\alpha\le 1. \tag{1.7.1} $$

The fractional integral with respect to $(dt)^\alpha$ was introduced by Jumarie [64], in order to study the fractional derivative of non-differentiable functions in modified Riemann-Liouville sense. Here, we are fully in Leibniz framework, that is to say $(dx)^\alpha$ denote finite increment in fractional sense. As a result, we shall be able to duplicate, in a straightforward manner, most of the known standard formulae by merely making the substitution $(dx)^\alpha \to dx$.

1.7.2 Fractional Riemann-Liouville Derivative

The fractional Riemann-Liouville derivative of $f(x)$ is defined as [77]

$$ {}_0D_x^\alpha\ f(x) = \begin{cases} \frac{d^n f}{dx^n}, & \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1} f(t)\,dt\,, & n-1<\alpha<n,\ n\in\mathbb{N}. \end{cases} \tag{1.7.2} $$

1.7.3 Modified Riemann-Liouville Derivative

Through the fractional Riemann-Liouville integral, Jumarie [62] proposed the modified Riemann-Liouville derivative of $f(x)$ as

$$ {}_0D_x^\alpha\ f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}(f(t)-f(0))\,dt\,,\ \ n-1<\alpha<n,\ n\in\mathbb{N} \tag{1.7.3} $$

1.7.4 Some Properties of Modified Riemann-Liouville Derivative

Here, some properties of modified Riemann-Liouville derivative are given which have been used in this work

(i) $df(x) = \frac{{}_0D_x^\alpha f(x)(dx)^\alpha}{\Gamma(1+\alpha)}, \quad \alpha > 0$

(ii) ${}_0D_t^\alpha(u(t)v(t)) = ({}_0D_t^\alpha u(t))v(t) + u(t)({}_0D_t^\alpha v(t)), \quad 0 < \alpha < 1$

(iii) ${}_0D_t^\alpha f(x(t)) = \frac{df}{dx}({}_0D_t^\alpha x(t)), \ 0 < \alpha < 1$, given $\frac{df}{dx}$ exists.

(iv) ${}_0D_x^\alpha x^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \ 0 < \alpha < 1, \quad x > 0$ and $\beta > -1$.

(v) $\int (dx)^\beta = x^\beta, \quad 0 < \beta \leq 1.$

(vi) $\Gamma(1+\beta)dx = d^\beta x.$

The above formulae and details thereof along with the scope of applications and limitations can be found in [62, 64].

1.7.5 Characteristic Method for Fractional order Differential Equations

It is well known that the method of characteristics is a very effective technique in solving partial differential equations. It applies to first-order equations, but generally this method is valid for any hyperbolic partial differential equation. With the modified Riemann-Liouville derivative, Jumarie [63] applied the Lagrange characteristic method to a class of fractional order partial differential equations, in which the time-fractional order equals to the space-fractional order. After that a more generalized fractional method of characteristics is presented by Wu [103]. Wu extended the method of characteristics for first order linear partial differential equations to a linear fractional differential equation of the form

$$a(x,t)\frac{\partial^\beta u(x,t)}{\partial x^\beta} + b(x,t)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = c(x,t), \tag{1.7.4}$$

where $0 < \alpha, \beta < 1$.

By expanding $u(x,t)$ as the fractional Jumarie- Taylors series of multivariate function [64], one obtains

$$du = \frac{\partial^{\beta} u(x,t)}{\Gamma(1+\beta)\partial x^{\beta}}(dx^{\beta}) + \frac{\partial^{\alpha} u(x,t)}{\Gamma(1+\alpha)\partial t^{\alpha}}(dt)^{\alpha}, \tag{1.7.5}$$

with $0 < \alpha, \beta < 1$. The generalised characteristic curves of equation (1.7.4) are

$$\frac{du}{ds} = c(x,t) \tag{1.7.6}$$

$$\frac{(dx^{\beta})}{\Gamma(1+\beta)ds} = a(x,t) \tag{1.7.7}$$

$$\frac{(dt)^{\alpha}}{\Gamma(1+\alpha)ds} = b(x,t), \tag{1.7.8}$$

where s is a parameter.

1.8 Problems to be Considered

Keeping in view the rich treasure and wide applicability of fractional differential equations in almost every field, we have in this thesis carried out the application of Lie group analysis for obtaining exact solutions to nonlinear fractional order partial differential equation and their systems also. In short, this thesis is devoted to applications of continuous symmetry groups to following physically important systems of fractional differential equations:

1. The space-time fractional Burgers–Poisson equation

2. The time fractional Potential Burgers' equation

3. The space-time fractional Potential Burgers' equation with variable coefficients

4. The time fractional Gardner equation

5. The fractional coupled KdV equation with variable coefficients

Chapter 2 deals with the study of following fractional order Burgers-Poisson equation(FBP equation)

$$u_t^{(\alpha)} - (u_x^{(2\beta)})_t^{\alpha} + u_x^{(\beta)} + uu_x^{(\beta)} - (3u_x^{(\beta)}u_x^{(2\beta)} + uu_x^{(3\beta)}) = 0, \tag{1.8.1}$$

where $x \in (0, \infty),\ \ t > 0,\ \ 0 < \alpha, \beta < 1.$

On carrying over the Lie group method to FBP equation, the groups of transformations admitted by the equation under consideration have been derived. Consequently, by using the symmetries involving arbitrary parameter, the FBP equation has been reduced to ODE which is further studied for group invariant solutions.

Chapter 3 is devoted to the study of following time fractional Potential Burgers' equation:

$$u_t^{(\alpha)} = Au_{xx} + B(u_x)^2, \quad x \in (0, \infty),\ t > 0,\ 0 < \alpha < 1, \tag{1.8.2}$$

where A and B are real constant parameters.

The time fractional Potential Burgers' equation is reduced to an ordinary differential equation of fractional order corresponding to the Erdélyi-Kober fractional derivative by using Lie classical symmetries. Further, an analytic solution is furnished by means of the Invariant Subspace Method.

In Chapter 4, a study has been made on the following space-time fractional Potential Burgers' (FPB) equation for invariance under continuous group of transformations via Lie classical approach:

$$u_t^{(\alpha)} = f(t)u_x^{(2\beta)} + g(t)\left(u_x^{(\beta)}\right)^2, \quad x \in (0, \infty),\ t > 0,\ 0 < \alpha, \beta < 1, \tag{1.8.3}$$

where $f(t)$ and $g(t)$ are arbitrary functions of t.

For the FPB equation, six-dimensional symmetries have been obtained and using the subalgebras of Lie algebras, it is shown that there are various group theoretic reductions of this equation depending on certain choices of infinitesimal generators. The reduced ODEs are investigated for group invariant solutions of FPB equation. The solutions obtained are new and involve arbitrary functions $f(t)$ and $g(t)$.

In Chapter 5, we study the classical symmetries of time fractional Gardner equation of the form:

$$u_t^{(\alpha)} = Auu_x + Bu^2u_x + u_{xxx}, \quad x \in (0,\infty),\ t > 0,\ 0 < \alpha < 1, \tag{1.8.4}$$

where A and B are real constant parameters. Certain new solutions of time fractional Gardner equation have been reported.

Chapter 6 is devoted to the space-time fractional coupled KdV equation with time dependent coefficients

$$u_t^{(\alpha)} + f(t)uu_x^{(\beta)} + g(t)vv_x^{(\beta)} + h(t)u_x^{(3\beta)} = 0, \tag{1.8.5}$$

$$v_t^{(\alpha)} + \delta(t)uv_x^{(\beta)} + k(t)v_x^{(3\beta)} = 0, \tag{1.8.6}$$

where $x \in (0,\infty),\ \ t > 0,\ \ 0 < \alpha, \beta < 1$.
Particular cases corresponding to certain specific values of the coefficients involved and those spatial forms for which the equation can be reduced to ODEs are presented.

Chapter 2

Group Invariant Solutions of Fractional Order Burgers-Poisson Equation

2.1 Introduction

In 2004, Fellnerand and Schmeiser detected that the Burgers-Poisson (BP) system

$$u_t + uu_x = \phi_x, \qquad \phi_{xx} = \phi + u, \tag{2.1.1}$$

where ϕ and u depend on $(t, x) \in R$, describes the unidirectional propagation of long waves in dispersive media. The BP system (2.1.1) can be easily replaced by the single BP equation

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{2.1.2}$$

Due to weaker dispersive effects for unidirectional water waves the BP equation turned out to be a better model equation compared to the Korteweg-de Vries (KdV) equation. Because of this property the BP equation has great importance in the field of mathematical physics and continuum mechanics. The authors in [30], presented few interesting behaviour patterns that BP equation exhibits, such as wave breaking in finite time, local existence results for smooth solutions and global existence result for weak entropy solutions. The Lie symmetries and group invariant solutions of equation (2.1.2) are reported in [98]. The numerical solutions of the BP equation have been worked out by Hizel and Kucukarslan [56] using the variational iteration method. To provide the phase velocity that arises in linear water wave theory, the BP equation is presented as an approximate model equation for water waves [67]. The large time behaviour of solutions to BP equation is presented by some authors in [31]. In recent times fractional differential equations have caught a remarkable attention of many researchers due to its extensive applications in many fields. Fractional versions of the well known equations of applied mathematics, such as the growth equation, diffusion equation, transport equation, Bloch equation, Schrödinger equation, etc., have produced many interesting solutions along with observable consequences. The application of Lie symmetries is one of the most effective techniques in solving nonlinear partial differential equations (PDEs). Only few researchers have applied the Lie group method on fractional differential equations. In 2010, the fractional Lie group method and the fractional characteristic method are proposed by Wu to solve anomalous diffusion equations [103, 104].

In this chapter, we present the application of classical Lie group method (section 1.5) to a fractional order Burgers-Poisson equation (FBP equation)

$$u_t^{(\alpha)} - (u_x^{(2\beta)})_t^{\alpha} + u_x^{(\beta)} + uu_x^{(\beta)} - (3u_x^{(\beta)}u_x^{(2\beta)} + uu_x^{(3\beta)}) = 0, \tag{2.1.3}$$

where $x \in (0, \infty)$, $t > 0$, $0 < \alpha, \beta < 1$. Equation (2.1.3) is obtained by replacing the first-order time and space derivatives by the fractional derivatives of order α and β in the classical Burgers-Poisson equation. The chapter has been organised as follows. The infinitesimals of the group of transformations which leaves the FBP equation invariant are obtained in section 2.2. Section 2.3 is entirely devoted to the determination of the reduced forms of ordinary differential equations (ODEs) and their exact solutions. Finally, the conclusion is given in Section 2.4

2.2 Symmetry Analysis of FBP Equation

In order to apply the Lie classical method to the FBP equation (2.1.3), we consider the Lie symmetries of the form

$$\frac{\tilde{x^{\beta}}}{\Gamma(1+\beta)} = \frac{x^{\beta}}{\Gamma(1+\beta)} + \epsilon\xi(x,t,u) + o(\epsilon^2) \tag{2.2.1}$$

$$\frac{\tilde{t^{\alpha}}}{\Gamma(1+\alpha)} = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \epsilon\tau(x,t,u) + o(\epsilon^2) \tag{2.2.2}$$

$$\tilde{u} = u + \epsilon\eta(x,t,u) + o(\epsilon^2), \tag{2.2.3}$$

where ϵ is the group parameter and ξ, τ and η are the infinitesimals of the transformations for the independent and dependent variables, respectively. It is therefore, necessary that this transformation leaves the set of solutions of equation (2.1.3) invariant. This yields an overdetermined linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$.
The associated Lie algebra of infinitesimal symmetries and its fractional third order prolongation is the set of vector fields of the form

$$V = \xi(x,t,u)\frac{\partial^{\beta}}{\partial x^{\beta}} + \tau(x,t,u)\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.2.4}$$

$$\begin{aligned} pr^{(3)}V = \xi(x,t,u)\frac{\partial^{\beta}}{\partial x^{\beta}} + \tau(x,t,u)\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \eta(x,t,u)\frac{\partial}{\partial u} + \eta^{t}\frac{\partial}{\partial u_t^{(\alpha)}} + \eta^{x}\frac{\partial}{\partial u_x^{(\beta)}} + \\ \eta^{xx}\frac{\partial}{\partial u_x^{(2\beta)}} + \eta^{xxt}\frac{\partial}{\partial (u_t^{(\alpha)})_x^{(2\beta)}} + \eta^{xxx}\frac{\partial}{\partial u_x^{(3\beta)}} \end{aligned} \tag{2.2.5}$$

Now for the invariance of equation (2.1.3) under equations (2.2.1)-(2.2.3), we must have

$$pr^{(3)}V([\Delta u])|_{[\Delta u]=0} = 0, \tag{2.2.6}$$

where $[\Delta u] = u_t^{(\alpha)} - (u_x^{(2\beta)})_t^{\alpha} + u_x^{(\beta)} + uu_x^{(\beta)} - (3u_x^{(\beta)}u_x^{(2\beta)} + uu_x^{(3\beta)})$,
or, equivalently

$$(\eta(u_x^{(\beta)} - u_x^{(3\beta)}) + \eta^{x}(1 + u - 3u_x^{(2\beta)}) + \eta^{t} - 3u_x^{(\beta)}\eta^{xx} - u\eta^{xxx} - \eta^{xxt})|_{([\Delta u])=0} = 0, \tag{2.2.7}$$

where $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ and η^{xxt} are extended (prolonged) infinitesimals given by the expressions

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)u_j$$

$$\eta_{i_1 i_2 \dots i_k}^{(k)} = D_{i_k}\eta_{i_1 i_2 \dots i_{k-1}}^{(k-1)} - (D_{i_k}\xi_j)u_{i_1, i_2, \dots, i_{k-1}, j},$$

where $i_l = 1, 2$ for $l = 1, 2$ with $k \geq 2$ and D_i is total derivative operator defined as

$$D_i = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \mathbf{u}_{ii_1i_2}\frac{\partial}{\partial u_{i_1i_2}} + \dots + u_{ii_1i_2\dots i_n}\frac{\partial}{\partial u_{i_1i_2\dots i_n}} + \dots,$$

with summation over repeated index.

Using the generalised fractional prolongation vector fields in equation (2.2.7) and equating the coefficient of various derivative terms to zero, we get the simplified set of determining equations as follows (for details, refer to Appendix-2A)

$$\tau_u = \xi_x^{(\beta)} = \xi_t^{(\alpha)} - \eta = 0 \tag{2.2.8}$$

$$\tau_x^{(\beta)} = \eta_x^{\beta} - \eta_x^{3\beta} = 0 \tag{2.2.9}$$

$$\xi_u = \xi_x^{(2\beta)} - 2(\eta_u)_x^{(\beta)} = 0 \tag{2.2.10}$$

$$\eta_{uu} = 0 \tag{2.2.11}$$

$$\eta_u + \tau_t^{(\alpha)} = \xi_x^{(\beta)} \tag{2.2.12}$$

$$\tau_t^{(\alpha)} = \xi_x^{(\beta)} \tag{2.2.13}$$

$$\xi_x^{(2\beta)} - (\eta_u)_x^{(\beta)} = 0 \tag{2.2.14}$$

$$2(\xi_x^{(\beta)})_t^{(\alpha)} - (\eta_u)_t^{(\alpha)} - 3\eta_x^{(\beta)} = 0 \tag{2.2.15}$$

$$\xi_x^{(3\beta)} - 2(\eta_u)_x^{(2\beta)} - \eta_x^{(2\beta)} + \xi_x^{(\beta)} + \tau_t^{(\alpha)} = 0 \tag{2.2.16}$$

$$\eta_t^{(\alpha)} + \eta_x^{(\beta)} - (\eta_x^{(2\beta)})_t^{\alpha} = 0 \tag{2.2.17}$$

Equations (2.2.8)–(2.2.17) enable us to derive the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\eta(x, t, u)$ as follows:

$$\xi = c\frac{t^{\alpha}}{\Gamma(1+\alpha)} + b, \tag{2.2.18}$$

$$\tau = a, \tag{2.2.19}$$

$$\eta = c, \tag{2.2.20}$$

where a, b, c are arbitrary constants.

Hence, the point symmetries under which the equation (2.1.3) is invariant can be spanned by the following three linearly independent infinitesimal generators:

$$V_1 = \frac{\partial^\alpha}{\partial t^\alpha} \tag{2.2.21}$$

$$V_2 = \frac{\partial^\beta}{\partial x^\beta} \tag{2.2.22}$$

$$V_3 = \frac{t^\alpha}{\Gamma(1+\alpha)} \frac{\partial^\beta}{\partial x^\beta} + \frac{\partial}{\partial u}. \tag{2.2.23}$$

Using these generators one can reduce the equation (2.1.3) to an ODE after getting the similarity variable by solving the characteristic equations (1.4.16) correspondig to each infinitesimal generator as given in section (1.4.1.4). In general one may obtain the reduced ODE from any linear combination of generators V_j, $j = 1, 2, 3$. Further, it may be noted that for $\alpha = \beta = 1$, the infinitesimals reported for the integer order model can be recovered [98]. With the help of equation (1.4.18) the commutator table for the vector fields in the Lie algebra can be constructed as follows: Lie group of local point transformations generated

Table 2.1: Commutator Table

	V_1	V_2	V_3
V_1	0	0	V_2
V_2	0	0	0
V_3	$-V_2$	0	0

by the vector field $V_i, i = 1, 2, 3$ and V, where $V = rV_1 + V_3$ is obtained by solving the system of ordinary differential equations

$$\frac{(d\tilde{x})^\beta}{\Gamma(1+\beta)d\epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) \tag{2.2.24}$$

$$\frac{(d\tilde{t})^\alpha}{\Gamma(1+\alpha)d\epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) \tag{2.2.25}$$

$$\frac{d\tilde{u}}{d\epsilon} = \eta(\tilde{x}, \tilde{t}, \tilde{u}), \tag{2.2.26}$$

with the initial conditions

$$\tilde{x}|_{\epsilon=0} = x \tag{2.2.27}$$

$$\tilde{t}|_{\epsilon=0} = t \tag{2.2.28}$$

$$\tilde{u}|_{\epsilon=0} = u \tag{2.2.29}$$

On solving the above equations, we get the following one parameter groups

$$g_1 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)} + \epsilon, u\right)$$

$$g_2 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)} + \epsilon, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right)$$

$$g_3 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)} + \epsilon\frac{t^\alpha}{\Gamma(1+\alpha)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u + \epsilon\right)$$

$$g : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)} + \epsilon\frac{t^\alpha}{\Gamma(1+\alpha)}, \frac{t^\alpha}{\Gamma(1+\alpha)} + r\epsilon, u + \epsilon\right)$$

2.3 Reduced Forms and Exact Solutions

In this section, we investigate some similarity transformations which reduce the FBP equation (2.1.3) to ordinary differential equation, further we obtain some exact solutions to the FBP equation (2.1.3) corresponding to the following infinitesimal generators

(i) V_1

(ii) V_2

(iii) V_3

(iv) $rV_1 + V_3$,

where r is an arbitrary nonzero constant parameter.

Theorem 2.3.1 *Under the invariants* $X(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$ *and* $\phi(X) = \frac{x^\beta}{\Gamma(1+\beta)} - \frac{t^\alpha}{\Gamma(1+\alpha)}u$ *the FBP equation (2.1.3) reduces to an ordinary differential equation*

$\phi(X)\phi'''(X) + 3\phi'(X)\phi''(X) - \phi(X)\phi'(X) - \phi(X) = 0$, which has the general solution in implicit form as $\int \frac{2\sqrt{3}|\phi|}{\sqrt{3\phi^4+8\phi^3+12c_1\phi^2-6c_2}} d\phi = X + c_3$, where c_1, c_2 and c_3 are arbitrary constants.

Proof: Consider the infinitesimal generator V_1, given by

$$V_1 = \frac{\partial^\alpha}{\partial t^\alpha}.$$

We find the resulting invariant solution by reducing equation (2.1.3) to a linear ordinary differential equation using differential invariants. The fractional characteristic equations for V_1 are

$$\frac{\frac{(dx)^\beta}{\Gamma(1+\beta)}}{0} = \frac{\frac{(dt)^\alpha}{\Gamma(1+\alpha)}}{1} = \frac{du}{0}. \tag{2.3.1}$$

From the fractional characteristic equations we obtain two functionally independent invariants as

$$X(x,t) = \frac{x^\beta}{\Gamma(1+\beta)}, \qquad \text{and} \qquad \phi(X) = u(x,t) \tag{2.3.2}$$

Now the solution of the fractional characteristic equations will be of the form $u(x,t) = \phi(X)$, therefore,

$$u(x,t) = \phi(\frac{x^\beta}{\Gamma(1+\beta)}). \tag{2.3.3}$$

Substituting this value of $u(x,t)$ in equation (2.1.3), we get the reduced third order nonlinear ordinary differential equation

$$\phi(X)\phi'''(X) + 3\phi'(X)\phi''(X) - \phi(X)\phi'(X) - \phi(X) = 0. \tag{2.3.4}$$

The general solution of the equation (2.3.4) in implicit form is obtained as

$$\int \frac{2\sqrt{3}|\phi|}{\sqrt{3\phi^4 + 8\phi^3 + 12c_1\phi^2 - 6c_2}} d\phi = X + c_3, \tag{2.3.5}$$

where c_1, c_2 and c_3 are arbitrary constants. This gives

$$u(x,t) = \phi(X), \tag{2.3.6}$$

where X and $\phi(X)$ are given by equations (2.3.2) and (2.3.5). In particular, taking $c_1 = 1$, $c_2 = 0$ a solution of FBP equation (2.1.3) is given by

$$u(x,t) = \frac{2}{3}[\sqrt{5}\sinh(\frac{x^\beta}{2\Gamma(1+\beta)}) - 2]. \tag{2.3.7}$$

Theorem 2.3.2 *The similarity transformations $u(x,t) = \frac{1}{r}[\frac{t^\alpha}{\Gamma(1+\alpha)} - \zeta(\theta)]$ along with the similarity variable $\theta(x,t) = r\frac{x^\beta}{\Gamma(1+\beta)} - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$ reduces the FBP equation (2.1.3) to nonlinear ordinary differential equation*

$$3r^2\zeta'(\theta)\zeta''(\theta) + r^2\zeta(\theta)\zeta'''(\theta) - \zeta(\theta)\zeta'(\theta) + r\zeta'(\theta) - 1 = 0.$$

Proof: On taking the infinitesimal generator

$$V_4 = rV_1 + V_3 = r\frac{\partial^\alpha}{\partial t^\alpha} + \frac{t^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\beta}{\partial x^\beta} + \frac{\partial}{\partial u}, \tag{2.3.8}$$

we obtain the invariants

$$\theta(x,t) = r\frac{x^\beta}{\Gamma(1+\beta)} - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \qquad \text{and} \qquad \zeta(\theta) = \frac{t^\alpha}{\Gamma(1+\alpha)} - ru(x,t) \tag{2.3.9}$$

and the reduced form of equation (2.1.3) as

$$3r^2\zeta'(\theta)\zeta''(\theta) + r^2\zeta(\theta)\zeta'''(\theta) - \zeta(\theta)\zeta'(\theta) + r\zeta'(\theta) - 1 = 0. \tag{2.3.10}$$

It can be easily seen that the differential equation (2.3.10) is invariant under the translation group

$$\theta^* = \theta + \epsilon, \quad \zeta^* = \zeta. \tag{2.3.11}$$

Hence, equation (2.3.10) can be reduced to the following second order differential equation

$$3r^2\phi^2(\zeta)\phi'(\zeta) + r^2\zeta(\phi^2(\zeta)\phi''(\zeta)) - \zeta\phi(\zeta) + r\phi(\zeta) - 1 = 0. \tag{2.3.12}$$

Further attempts in search of invariant one-parameter Lie groups of transformation for equation (2.3.10) do not yield any nontrivial group.

Theorem 2.3.3 *Under the group of point transformations $X(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$ and $\phi(X) = \frac{x^\beta}{\Gamma(1+\beta)} - \frac{t^\alpha}{\Gamma(1+\alpha)}u$ the FBP equation (2.1.3) reduces to an ordinary differential equation $\phi'(X) = 1$, which has the general solution as $u(x,t) = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\frac{x^\beta}{t^\alpha} - K\frac{\Gamma(1+\alpha)}{t^\alpha} - 1$, where K is an arbitrary constant.*

Proof: In this case, we study the infinitesimal generator

$$V_3 = \frac{t^\alpha}{\Gamma(1+\alpha)} \frac{\partial^\beta}{\partial x^\beta} + \frac{\partial}{\partial u}. \tag{2.3.13}$$

The following invariants can be derived easily

$$X(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}, \qquad \phi(X) = \frac{x^\beta}{\Gamma(1+\beta)} - \frac{t^\alpha}{\Gamma(1+\alpha)} u, \tag{2.3.14}$$

and the reduced form of equation (2.1.3) is

$$\phi'(X) = 1. \tag{2.3.15}$$

This yields the solution

$$u(x,t) = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)} \frac{x^\beta}{t^\alpha} - K \frac{\Gamma(1+\alpha)}{t^\alpha} - 1. \tag{2.3.16}$$

Theorem 2.3.4 *Under the transformations $X(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$ and $\phi(X) = u(x,t)$ the FBP equation (2.1.3) reduces to an ordinary differential equation of first order $\phi'(X) = 0$, which gives constant solution $u(x,t) = k$.*

Proof: In this case we obtain an invariant solution of equation (2.1.3) by using the infinitesimal generator

$$V_2 = \frac{\partial^\beta}{\partial x^\beta}. \tag{2.3.17}$$

Here, the fractional characteristic equations give the invariants

$$X(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}, \qquad \phi(X) = u(x,t). \tag{2.3.18}$$

This gives

$$u(x,t) = \phi(\frac{t^\alpha}{\Gamma(1+\alpha)}). \tag{2.3.19}$$

Using the new form of $u(x,t)$ in equation (2.1.3), it reduces to an ordinary differential equation of first order

$$\phi'(X) = 0. \tag{2.3.20}$$

which leads to the constant solution

$$u(x,t) = k. \tag{2.3.21}$$

2.4 Discussion

The Lie symmetry analysis is the most effective and important analytic approach to obtain exact solutions for nonlinear differential equations of integer order. This chapter shows the effectiveness of the method in solving fractional order differential equations. In particular, the fractional Lie group method has been effectively applied on a nonlinear fractional Burgers-Poisson equation. For various infinitesimal generators the Burgers-Poisson equation has been reduced into some ordinary differential equations by using the method of differential invariants. Further, utilising the one-dimensional Lie symmetry generators admitted by the FBP equation (2.1.3) some group invariant solutions of FBP equation (2.1.3) have also been provided. The method can be applied on various other nonlinear fractional order partial differential equations.

Appendix–2A

The extended infinitesimals $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ and η^{xxt} can be easily obtained as

$$\eta^t = \eta_t^{(\alpha)} + u_t^{(\alpha)}\eta_u - (\xi_t^{(\alpha)} + u_t^{(\alpha)}\xi_u)u_x^{(\beta)} - (\tau_t^{(\alpha)} + u_t^{(\alpha)}\tau_u)u_t^{(\alpha)},$$

$$\eta^x = \eta_x^{(\beta)} + u_x^{(\beta)}\eta_u - (\xi_x^{(\beta)} + u_x^{(\beta)}\xi_u)u_x^{(\beta)} + (\tau_x^{(\beta)} + u_x^{(\beta)}\tau_u)u_t^{(\alpha)},$$

$$\begin{aligned}\eta^{xx} = \eta_x^{(2\beta)} + u_x^{(\beta)}(\eta_u)_x^{(\beta)} - (\xi_x^{(2\beta)} + u_x^{(\beta)}(\xi_u)_x^{(\beta)})u_x^{(\beta)} + (\tau_x^{(2\beta)} + u_x^{(\beta)}(\tau_u)_x^{(\beta)})u_t^{(\alpha)} - \\ 2(\tau_x^{(\beta)} + u_x^{(\beta)}\tau_u)(u_x^{(\beta)})_t^{(\alpha)} + u_x^{(2\beta)}(\eta_u - u_x^{(\beta)}\xi_u - u_t^{(\alpha)}\tau_u) - 2(\xi_x^{(\beta)} + u_x^{(\beta)}\xi_u)u_x^{(2\beta)} + \\ [(\eta_u)_x^{(\beta)} + u_x^{(\beta)}\eta_{uu} - ((\xi_u)_x^{(\beta)} + u_x^{(\beta)}\xi_{uu})u_x^{(\beta)} - ((\tau_u)_x^{(\beta)} + u_x^{(\beta)}\tau_{uu})u_t^{(\alpha)}]u_x^{(\beta)},\end{aligned}$$

$$\begin{aligned}\eta^{xxx} = \eta_x^{(3\beta)} + (3(\eta_u)_x^{(2\beta)} - \xi_x^{(3\beta)})u_x^{(\beta)} - \tau_x^{(3\beta)}u_t^{(\alpha)} - 3(\tau_u)_x^{(2\beta)}u_t^{(\alpha)}u_x^{(\beta)} + \\ (3(\eta_{uu})_x^{(\beta)} - (\xi_u)_x^{(2\beta)})(u_x^{(\beta)})^2 - 3(\tau_{uu})_x^{(\beta)}u_t^{(\alpha)}(u_x^{(\beta)})^2 + ((\eta_{uuu}) - 3(\xi_{uu})_x^{(\beta)})(u_x^{(\beta)})^3 - \\ \tau_{uuu}u_t^{(\alpha)}(u_x^{(\beta)})^3 - \xi_{uuu}(u_x^{(\beta)})^4 - 3\tau_{xx}(u_x^{(\beta)})_t^{(\alpha)} + (3(\eta_u)_x^{(\beta)} - \xi_x^{(2\beta)})u_x^{(2\beta)} + \\ 3((\eta_{uu}) - 3(\xi_u)_x^{(\beta)})u_x^{(\beta)}u_x^{(2\beta)} - 3(\tau_u)_x^{(\beta)}u_t^{(\alpha)}u_x^{(2\beta)} - 6(\tau_u)_x^{(\beta)}u_x^{(\beta)}(u_x^{(\beta)})_t^{(\alpha)} - 6\xi_{uu}u_x^{(2\beta)}(u_x^{(\beta)})^2 - \\ 3\tau_{uu}(u_x^{(\beta)})^2(u_x^{(\beta)})_t^{(\alpha)} - 3\tau_{uu}u_x^{(2\beta)}u_x^{(\beta)}u_t^{(\alpha)} - 3\xi_u(u_x^{(2\beta)})^2 - 3\tau_u u_x^{(2\beta)}(u_x^{(\beta)})_t^{(\alpha)} - 3\tau_x(u_x^{(2\beta)})_t^{(\alpha)} + \\ (\eta_u - 3\xi_x^{(\beta)})(u_x^{(3\beta)}) - 4\xi_u(u_x^{(3\beta)})u_x^{(\beta)} - \tau_u(u_x^{(3\beta)})u_t^{(\alpha)} - 3\tau_u(u_x^{(\beta)})(u_x^{(2\beta)})_t^{(\alpha)}\end{aligned}$$

$$\begin{aligned}\eta^{xxt} = (\eta_x^{(2\beta)})_t^{(\alpha)} + ((\eta_u)_x^{(2\beta)} - (\tau_x^{(2\beta)})_t^{(\alpha)})u_t^{(\alpha)} + (2((\eta_u)_x^{(\beta)})_t^{(\alpha)} - (\xi_x^{(2\beta)})_t^{(\alpha)})u_x^{(\beta)} + \\ ((\eta_{uu})_t^{(\alpha)} - 2((\xi_u)_x^{(\beta)})_t^{(\alpha)})(u_x^{(2\beta)})^2 + (2(\eta_{uu})_x^{(\beta)} - (\xi_u)_x^{(2\beta)} - 2((\tau_u)_x^{(\beta)})_t^{(\alpha)})u_x^{(\beta)}u_t^{(\alpha)} - \\ 2(\tau_{uu})_x^{(\beta)}u_x^{(\beta)}(u_t^{(\alpha)})^2 - (\xi_{uu})_t^{(\alpha)}(u_x^{(\beta)})^3 + ((\eta_{uuu}) - 2(\xi_{uu})_x^{(\beta)} - (\tau_{uu})_t^{(\alpha)})(u_x^{(\beta)})^2u_t^{(\alpha)} - \\ \tau_{uuu}(u_t^{(\alpha)})^2(u_x^{(\beta)})^2 - \xi_{uuu}(u_x^{(\beta)})^3u_t^{(\alpha)} + ((\eta_u)_t^{(\alpha)} - 2((\xi)_x^{(\beta)})_t^{(\alpha)})u_x^{(2\beta)} + \\ (2(\eta_u)_x^{(\beta)} - \xi_x^{(2\beta)} - 2(\tau_x^{(\beta)})_t^{(\alpha)})(u_x^{(\beta)})_t^{(\alpha)} - \tau_x^{(2\beta)})u_t^{(2\alpha)} - 4(\tau_u)_x^{(\beta)}(u_x^{(\beta)})_t^{(\alpha)}u_t^{(\alpha)} - \\ 2(\tau_u)_x^{(\beta)}u_t^{(2\alpha)}u_x^{(\beta)} - 3(\xi_u)_t^{(\alpha)}u_x^{(2\beta)}u_x^{(\beta)} - (\tau_u)_x^{(2\beta)}(u_t^{(\alpha)})^2\end{aligned}$$

Using the expressions for $\eta^t, \eta^x, \eta^{xx}, \eta^{xxx}$ and η^{xxt} in equation (2.2.7), we eventually arrive at the following:

$$
\begin{gathered}
\eta(u_x^{(\beta)}-u_x^{(3\beta)})+(1+u-3u_x^{(2\beta)})(\eta_x^{(\beta)}+u_x^{(\beta)}\eta_u-(\xi_x^{(\beta)}+u_x^{(\beta)}\xi_u)u_x^{(\beta)}+(\tau_x^{(\beta)}+u_x^{(\beta)}\tau_u)u_t^{(\alpha)})+\\
\eta_t^{(\alpha)}+u_t^{(\alpha)}\eta_u-(\xi_t^{(\alpha)}+u_t^{(\alpha)}\xi_u)u_x^{(\beta)}-(\tau_t^{(\alpha)}+u_t^{(\alpha)}\tau_u)u_t^{(\alpha)}-\\
3u_x^{(\beta)}(\eta_x^{(2\beta)}+u_x^{(\beta)}(\eta_u)_x^{(\beta)}-(\xi_x^{(2\beta)}+u_x^{(\beta)}(\xi_u)_x^{(\beta)})u_x^{(\beta)}+(\tau_x^{(2\beta)}+u_x^{(\beta)}(\tau_u)_x^{(\beta)})u_t^{(\alpha)})-\\
6u_x^{(\beta)}((\tau_x^{(\beta)}+u_x^{(\beta)}\tau_u)(u_x^{(\beta)})_t^{(\alpha)}+u_x^{(2\beta)}(\eta_u-u_x^{(\beta)}\xi_u-u_t^{(\alpha)}\tau_u)-2(\xi_x^{(\beta)}+u_x^{(\beta)}\xi_u)u_x^{(2\beta)})+\\
3u_x^{(\beta)}([(\eta_u)_x^{(\beta)}+u_x^{(\beta)}\eta_{uu}-((\xi_u)_x^{(\beta)}+u_x^{(\beta)}\xi_{uu})u_x^{(\beta)}-((\tau_u)_x^{(\beta)}+u_x^{(\beta)}\tau_{uu})u_t^{(\alpha)}]u_x^{(\beta)})-\\
u(\eta_x^{(3\beta)}+(3(\eta_u)_x^{(2\beta)}-\xi_x^{(3\beta)})u_x^{(\beta)}-\tau_x^{(3\beta)}u_t^{(\alpha)}-3(\tau_u)_x^{(2\beta)}u_t^{(\alpha)}u_x^{(\beta)})+\\
u((3(\eta_{uu})_x^{(\beta)}-(\xi_u)_x^{(2\beta)})(u_x^{(\beta)})^2-3(\tau_{uu})_x^{(\beta)}u_t^{(\alpha)}(u_x^{(\beta)})^2+((\eta_{uuu})-3(\xi_{uu})_x^{(\beta)})(u_x^{(\beta)})^3)-\\
u(\tau_{uuu}u_t^{(\alpha)}(u_x^{(\beta)})^3-\xi_{uuu}(u_x^{(\beta)})^4-3\tau_{xx}(u_x^{(\beta)})_t^{(\alpha)}+(3(\eta_u)_x^{(\beta)}-\xi_x^{(2\beta)})u_x^{(2\beta)})+\\
u(3((\eta_{uu})-3(\xi_u)_x^{(\beta)})u_x^{(\beta)}u_x^{(2\beta)}-3(\tau_u)_x^{(\beta)}u_t^{(\alpha)}u_x^{(2\beta)}-6(\tau_u)_x^{(\beta)}u_x^{(\beta)}(u_x^{(\beta)})_t^{(\alpha)}-6\xi_{uu}u_x^{(2\beta)}(u_x^{(\beta)})^2)-\\
u(3\tau_{uu}(u_x^{(\beta)})^2(u_x^{(\beta)})_t^{(\alpha)}-3\tau_{uu}u_x^{(2\beta)}u_x^{(\beta)}u_t^{(\alpha)}-3\xi_u(u_x^{(2\beta)})^2-3\tau_u u_x^{(2\beta)}(u_x^{(\beta)})_t^{(\alpha)}-3\tau_x(u_x^{(2\beta)})_t^{(\alpha)})+\\
u((\eta_u-3\xi_x^{(\beta)})(u_x^{(3\beta)})-4\xi_u(u_x^{(3\beta)})u_x^{(\beta)}-\tau_u(u_x^{(3\beta)})u_t^{(\alpha)}-3\tau_u(u_x^{(\beta)})(u_x^{(2\beta)})_t^{(\alpha)})-(\eta_x^{(2\beta)})_t^{(\alpha)}+\\
((\eta_u)_x^{(2\beta)}-(\tau_x^{(2\beta)})_t^{(\alpha)})u_t^{(\alpha)}+(2((\eta_u)_x^{(\beta)})_t^{(\alpha)}-(\xi_x^{(2\beta)})_t^{(\alpha)})u_x^{(\beta)}+\\
((\eta_{uu})_t^{(\alpha)}-2((\xi_u)_x^{(\beta)})_t^{(\alpha)})(u_x^{(2\beta)})^2+(2(\eta_{uu})_x^{(\beta)}-(\xi_u)_x^{(2\beta)}-2((\tau_u)_x^{(\beta)})_t^{(\alpha)})u_x^{(\beta)}u_t^{(\alpha)}-\\
2(\tau_{uu})_x^{(\beta)}u_x^{(\beta)}(u_t^{(\alpha)})^2-(\xi_{uu})_t^{(\alpha)}(u_x^{(\beta)})^3+((\eta_{uuu})-2(\xi_{uu})_x^{(\beta)}-(\tau_{uu})_t^{(\alpha)})(u_x^{(\beta)})^2u_t^{(\alpha)}-\\
\tau_{uuu}{u_t^{(\alpha)}}^2(u_x^{(\beta)})^2-\xi_{uuu}(u_x^{(\beta)})^3u_t^{(\alpha)}+((\eta_u)_t^{(\alpha)}-2((\xi)_x^{(\beta)})_t^{(\alpha)})u_x^{(2\beta)}+\\
(2(\eta_u)_x^{(\beta)}-\xi_x^{(2\beta)}-2(\tau_x^{(\beta)})_t^{(\alpha)})(u_x^{(\beta)})_t^{(\alpha)}-\tau_x^{(2\beta)})u_t^{(2\alpha)}-4(\tau_u)_x^{(\beta)}(u_x^{(\beta)})_t^{(\alpha)}u_t^{(\alpha)}-\\
2(\tau_u)_x^{(\beta)}u_t^{(2\alpha)}u_x^{(\beta)}-3(\xi_u)_t^{(\alpha)}u_x^{(2\beta)}u_x^{(\beta)}-(\tau_u)_x^{(2\beta)}(u_t^{(\alpha)})^2|_{([\Delta u])=0}=0 \quad \text{(2A-1)}
\end{gathered}
$$

On equating the coefficients of different differentials equal to zero, we obtained a set of determining equations as (2.2.8)–(2.2.17)

Chapter 3

Symmetry Analysis of Time Fractional Potential Burgers' Equation

3.1 Introduction

In 2007, Gazizov investigated the continuous point transformation groups of some fractional differential equations and proposed some prolongation formulae, where the author assumed the existence of both, the fractional derivative as well as the integer order derivative. In this chapter by means of Lie group method we consider the following time fractional potential Burgers' equation of the form

$$u_t^{(\alpha)} = Au_{xx} + B(u_x)^2, \quad x \in (0, \infty),\ t > 0,\ 0 < \alpha < 1, \tag{3.1.1}$$

where A and B are real constant parameters.

Lie point symmetries of time fractional potential Burgers' equation are presented. Using these symmetries the time fractional potential Burgers' equation (3.1.1) has been transformed into an ordinary differential equation of fractional order corresponding to the Erdélyi-Kober fractional derivative [69]. Further, an analytic solution is furnished by means of the Invariant Subspace Method.

In the next section, we deduce the Lie symmetries of the equation (3.1.1). Section 3.3 is devoted in finding some group invariant solutions on solving the reduced forms of ODEs associated with some basic fields of sub algebras. Some other exact solutions have been investigated in section 3.4 using Invariant Subspace method. Finally, in the last section we make some concluding remarks.

3.2 Lie Symmetries

Herein, we investigate the symmetries of time fractional Potential Burgers' equation (3.1.1). We assume that eqn. (3.1.1) admits the Lie symmetries of the form

$$\tilde{x} = x + \epsilon\xi(x,t,u) + o(\epsilon^2) \tag{3.2.1}$$

$$\tilde{t} = t + \epsilon\tau(x,t,u) + o(\epsilon^2) \tag{3.2.2}$$

$$\tilde{u} = u + \epsilon\eta(x,t,u) + o(\epsilon^2), \tag{3.2.3}$$

where ϵ is the group parameter and ξ, τ and η are the infinitesimals of the transformations for the independent and dependent variables, respectively. In other words if u is solution of equation (3.1.1), then $\tilde{u}$ is also a solution.
The associated Lie algebra of infinitesimal symmetries of equation (3.1.1) is then the vector field of the form

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{3.2.4}$$

The fractional second order prolongation of (3.1.1) is

$$\begin{aligned} pr^{(2)}V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \\ \eta^0_\alpha\frac{\partial}{\partial u_t^{(\alpha)}} + \eta^{tt}\frac{\partial}{\partial u_{tt}} + \eta^{xx}\frac{\partial}{\partial u_{xx}}. \end{aligned} \tag{3.2.5}$$

Invoking the invariance criterion as explained in chapter 1, the following relation from the coefficients of the first order of ϵ is deduced:

$$(\eta^0_\alpha - 2Bu_x\eta^x - A\eta^{xx})|_{([\Delta u])=0} = 0, \tag{3.2.6}$$

where η^x, η^0_α and η^{xx} are extended infinitesimals acting on enlarged space corresponding to u_x, u^α_t and u_{xx}. The method for determining the symmetry group of equation (3.1.1) mainly consists of finding the infinitesimals ξ, τ and η, which are functions of x, t and u. The general solution of equation (3.2.6) provides the infinitesimal elements ξ, τ and η for which the equation (3.1.1) possesses Lie symmetry. Using the expressions for η^x, η^0_α and η^{xx} (for these expressions refer to Appendix-3A) in equation (3.2.6) and equating the coefficients of different differentials equal to zero, we obtained a set of determining equations as follows:

$$\tau_u = 0 \tag{3.2.7}$$

$$\tau_x = 0 \tag{3.2.8}$$

$$\xi_u = 0 \tag{3.2.9}$$

$$\binom{\alpha}{n}\partial_t^n\eta - \binom{\alpha}{n+1}D_t^{n+1}(\tau) = 0, \quad n = 1, 2, 3, ... \tag{3.2.10}$$

$$2\xi_x - \alpha\tau_u = 0 \tag{3.2.11}$$

$$2B\xi_x - B\alpha\tau_u - B\tau_u = 0 \tag{3.2.12}$$

$$D_t^n(\xi) = 0, \quad n = 1, 2, 3, ... \tag{3.2.13}$$

On solving the above equations (3.2.7-3.2.13), we obtain the infinitesimals as

$$\xi = c_1 x + c_2 \tag{3.2.14}$$

$$\tau = \frac{2c_1 t}{\alpha} \tag{3.2.15}$$

$$\eta = c_3 \tag{3.2.16}$$

where c_1,c_2,c_3 are arbitrary parameters. The point symmetry generators admitted by the equation (3.1.1) are given by

$$V_1 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} \tag{3.2.17}$$

$$V_2 = \frac{\partial}{\partial x} \tag{3.2.18}$$

$$V_3 = \frac{\partial}{\partial u} \tag{3.2.19}$$

Hence, the infinitesimal operator (3.2.4) becomes $V = (c_1x+c_2)\frac{\partial}{\partial x}+\frac{2c_1t}{\alpha}\frac{\partial}{\partial t}+c_3\frac{\partial}{\partial u}$. Further, these infinitesimal generators (3.2.17-3.2.19) can be used to determine a three parameters fractional Lie group of point transformations acting on (x, t, u)-space which is fewer than those for the standard Burgers' equation [84]. It can be verified easily that the set $\{V_1, V_2, V_3\}$ forms a three dimensional Lie algebra under the Lie bracket $[X, Y] = XY - YX$ and its commutator table is as given below:

Table 3.1: Commutator Table

	V_1	V_2	V_3
V_1	0	-V_2	0
V_2	V_2	0	0
V_3	0	0	0

Further, from the commutator table it can be seen that V_3 forms a solvable subalgebra. Also V_3 is the centre of the three dimensional Lie algebra as it commutes with every element of the Lie algebra. The group transformation generated by the infinitesimal generators $V_i, i = 1, 2, 3$ is obtained by solving the system of ordinary differential equations

$$\frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) \tag{3.2.20}$$

$$\frac{d\tilde{t}}{d\epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) \tag{3.2.21}$$

$$\frac{d\tilde{u}}{d\epsilon} = \eta(\tilde{x}, \tilde{t}, \tilde{u}) \tag{3.2.22}$$

with the initial conditions

$$\tilde{x}|_{\epsilon=0} = x \tag{3.2.23}$$

$$\tilde{t}|_{\epsilon=0} = t \tag{3.2.24}$$

$$\tilde{u}|_{\epsilon=0} = u. \tag{3.2.25}$$

Exponentiating the infinitesimal symmetries of equation (3.1.1), we get the one parameter groups $g_i(\epsilon)$ generated by $V_i, i = 1, 2, 3$

$$g_1 : (x, t, u) \to (e^{\epsilon}x, e^{\frac{2}{\alpha}}t, u) \tag{3.2.26}$$

$$g_2 : (x, t, u) \to (x + \epsilon, t, u) \tag{3.2.27}$$

$$g_3 : (x, t, u) \to (x, t, u + \epsilon) \tag{3.2.28}$$

Now, since g_i is a symmetry, if $u = f(x, t)$ is a solution of equation (3.1.1) the following u_i are also solutions of eqn. (3.1.1)

$$u_1 = f(e^{\epsilon}x, e^{\frac{2}{\alpha}}t) \tag{3.2.29}$$

$$u_2 = f(x + \epsilon, t) \tag{3.2.30}$$

$$u_3 = f(x, t) - \epsilon \tag{3.2.31}$$

3.3 Reduction to ODE

Herein, the time fractional potential Burgers' equation (3.1.1) has been reduced into an ODE with the Erdélyi-Kober fractional differential operator. For the infinitesimal generator V_1 the characteristic equations are

$$\frac{dx}{x} = \frac{\alpha dt}{2t} = \frac{du}{0}, \tag{3.3.1}$$

which give the invariants as $u(x, t) = f(z)$, $z = xt^{\frac{-\alpha}{2}}$. Corresponding to these invariants equation (3.1.1) can be reduced into an ODE of fractional order. The result has been summarized in the following theorem :

Theorem 3.3.1 *The similarity transformation $u(x, t) = f(z)$ along with the similarity variable $z = xt^{\frac{-\alpha}{2}}$ reduces the time fractional potential Burgers' equation (3.1.1) to the ordinary differential equation of fractional order of the form*

$$(P^{1-\alpha,\alpha}_{\frac{2}{\alpha}}f)(z) = A\frac{d^2f}{dz^2} + B(\frac{df}{dz})^2 \tag{3.3.2}$$

with the Erdélyi-Kober fractional differential operator [69]

$$(P_\delta^{\tau,\alpha} f)(z) = \prod_{j=0}^{m-1}(\tau + j - \frac{1}{\delta} z\frac{d}{dz})(K_\delta^{\tau+\alpha,m-\alpha} f)(z),\ z > 0,\ \delta > 0,\ \alpha > 0, \quad (3.3.3)$$

$$m = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N}. \end{cases}, \quad \textit{where}$$

$$(K_\delta^{\tau,\alpha} f)\ (z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_0^\infty (\nu - 1)^{\alpha-1} \nu^{-(\tau+\alpha)} f(z\nu^{\frac{1}{\delta}})\, d\nu\,, & \alpha > 0; \\ f(z), & \alpha = 0 \end{cases} \quad (3.3.4)$$

is the Erdélyi-Kober fractional integral operator.

Proof: Let $n - 1 < \alpha < n$, $n = 1, 2, 3, ...$ then the Riemann-Liouville fractional derivative for the similarity transformation $u(x, t) = f(z)$ with the similarity variable $z = xt^{\frac{-\alpha}{2}}$ becomes
$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} [\frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t - s)^{n-\alpha-1} f(xs^{\frac{-\alpha}{2}})\, ds].$
Let $\nu = \frac{t}{s}$. Then the above equation can be written as
$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} [t^{n-\alpha} \frac{1}{\Gamma(n-\alpha)} \int\limits_1^\infty (\nu - 1)^{n-\alpha-1} \nu^{-(n-\alpha+1)} f(z\nu^{\frac{\alpha}{2}})\, d\nu].$
Following the definition of the Erdélyi-Kober fractional integral operator given in equation (3.3.4), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} [t^{n-\alpha} (K_{\frac{2}{\alpha}}^{1,n-\alpha} f)(z)]. \quad (3.3.5)$$

In order to simplify the right hand side of equation (3.3.5), consider the relation $z = xt^{\frac{-\alpha}{2}}$, $f \in C^1(0, \infty)$,

$$\begin{aligned} t\frac{\partial}{\partial t} f(z) &= tx(-\frac{\alpha}{2})t^{-\frac{\alpha}{2}-1} f'(z) \\ &= -\frac{\alpha}{2} z \frac{d}{dz} f(z) \end{aligned}$$

and thus, we get

$$\begin{aligned} \frac{\partial^n}{\partial t^n} [t^{n-\alpha} (K_{\frac{2}{\alpha}}^{1,n-\alpha} f)(z)] &= \frac{\partial^{n-1}}{\partial t^{n-1}} [\frac{\partial}{\partial t} (t^{n-\alpha} (K_{\frac{2}{\alpha}}^{1,n-\alpha} f)(z))] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}} [t^{n-\alpha-1} (n - \alpha - \frac{\alpha}{2} z \frac{d}{dz} (K_{\frac{2}{\alpha}}^{1,n-\alpha} f)(z))]. \end{aligned}$$

Repeating the similar procedure for $n-1$ times, we have

$$\frac{\partial^n}{\partial t^n}[t^{n-\alpha}(K_{\frac{2}{\alpha}}^{1,n-\alpha}f)(z)] = t^{-\alpha}\prod_{j=0}^{n-1}(1-\alpha+j-\frac{\alpha}{2}z\frac{d}{dz})(K_{\frac{2}{\alpha}}^{1,n-\alpha}f)(z). \tag{3.3.6}$$

Now using the definition of Erdélyi-Kober fractional differential operator given in equation (3.3.3), the above equation can be written as

$$\frac{\partial^n}{\partial t^n}[t^{n-\alpha}(K_{\frac{2}{\alpha}}^{1,n-\alpha}f)(z)] = t^{-\alpha}(P_{\frac{2}{\alpha}}^{1-\alpha,\alpha}f)(z). \tag{3.3.7}$$

Thus, an expression for the time fractional derivative is

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha}(P_{\frac{2}{\alpha}}^{1-\alpha,\alpha}f)(z). \tag{3.3.8}$$

Continuing further we find that the time fractional potential Burgers' equation (3.1.1) reduces to an ordinary differential equation of fractional order

$$(P_{\frac{2}{\alpha}}^{1-\alpha,\alpha}f)(z) = A\frac{d^2f}{dz^2} + B(\frac{df}{dz})^2. \tag{3.3.9}$$

As the order $0 < \alpha < 1$ of the reduced equation is arbitrary, there is no existing method to solve the above differential equation of fractional order in general. However, for some special cases, such as the initial value problems and the linear equations, the solutions can be furnished by the power series method with Mittag-Leffler function and Wright and the generalised Wright functions [18]. In particular, when $B = 0$, two independent solutions of equation (3.1.1) can be drived as $W(-\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}}; -\frac{\alpha}{2}, 1)$ and $W(\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}}; -\frac{\alpha}{2}, 1)$, where $W(z;\lambda,\mu)$ is the Wright function [102] given by $W(z;\lambda,\mu) = \sum_{i=0}^{\infty}\frac{z^i}{i!\Gamma(\lambda i+\mu)}$. Consequently, the group invariant solution of equation (3.1.1) when $B = 0$ has the form

$$u(x,t) = K_1 W(-\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}}; -\frac{\alpha}{2}, 1) + K_2 W(\frac{xt^{-\frac{\alpha}{2}}}{\sqrt{A}}; -\frac{\alpha}{2}, 1), \tag{3.3.10}$$

where K_1 and K_2 are arbitrary parameters.

3.4 Some Exact Solutions of Time Fractional Potential Burgers' Equation by the Invariant Subspace Method.

The invariant subspace method (as defined in section 1.6) has been utilized in order to discover exact solutions of equation (3.1.1).
For equation (3.1.1),

$$F[u] = Au_{xx} + B{u_x}^2. \tag{3.4.1}$$

We have the space $W_3 = \langle 1, x, x^2 \rangle$ as invariant under $F[u]$, since

$$\begin{aligned} F[C_1 + C_2x + C_3x^2] &= 2C_3A + B(C_2 + 2C_3x)^2 \\ &= b_1 + b_2x + b_3x^2 \in W_3 \end{aligned}$$

where b_1,b_2 and b_3 are arbitrary constants given by

$$\begin{gathered} 2C_3A + BC_2^2 = b_1 \\ 4BC_2C_3 = b_2, \text{ and} \\ 4BC_3^2 = b_3. \end{gathered}$$

This allows us to consider an exact solution of equation (3.1.1) as

$$u(x,t) = a_1(t) + a_2(t)x + a_3(t)x^2. \tag{3.4.2}$$

Substituting the value of $u(x,t)$ from equation (3.4.2) into the equation (3.1.1) and equating the coefficients of x^j, $j = 0, 1, 2$, we get the following system of fractional differential equations

$$\frac{d^\alpha a_3(t)}{dt^\alpha} = 4B(a_3(t))^2, \tag{3.4.3}$$

$$\frac{d^\alpha a_2(t)}{dt^\alpha} = 4Ba_2(t)a_3(t) \tag{3.4.4}$$

and

$$\frac{d^\alpha a_1(t)}{dt^\alpha} = 2Aa_3(t) + B(a_2(t))^2. \tag{3.4.5}$$

Equations (3.4.3-3.4.5) can be readily solved to yield

$$-\frac{1}{a_3(t)} = \frac{2}{\Gamma(1+\alpha)} \int B dt^{\alpha} + s_1, \tag{3.4.6}$$

$$\log a_2(t) = \frac{4}{\Gamma(1+\alpha)} \int a_3(t) B dt^{\alpha} + s_2 \tag{3.4.7}$$

and

$$a_1(t) = \frac{1}{\Gamma(1+\alpha)} [\int (2a_3(t)A + (a_2(t))^2 B) dt^{\alpha} + s_3], \tag{3.4.8}$$

where s_1,s_2 and s_3 are arbitrary constants.
Using equation (3.4.2) and equations (3.4.6-3.4.8) one can easily obtain an exact solution of equation (3.1.1).

3.5 Discussion

The application of Lie symmetry method has been performed on a time fractional potential Burgers' equation (3.1.1) and the Lie point symmetries has been drived. The Lie symmetry analysis shows that the underlying symmetry algebra of the equation (3.1.1) is three dimensional unlike the six dimensional Lie algebra for standard potential Burgers' equation. The reduction of dimension in the symmetry algebra is due to the fact that the time fractional equation is not invariant under time translation symmetry. It is appropriate to mention here that the fractional order significantly affects the properties of the equation. The main reason is that the fractional order $0 < \alpha < 1$ is an arbitrary parameter in the studied fractional model. Using the Lie point symmetries, It has been shown that the equation can be transformed into an ODE of fractional order with Erdélyi-Kober fractional derivative. At last, some exact solutions to the time fractional potential Burgers' equation (3.1.1) are furnished by means of the fractional invariant subspace method.

Appendix–3A

The generalised fractional prolongation vector fields, η^x, η^0_α and η^{xx} are given by

$$\eta^x = \eta_x + u_x\eta_u - (\xi_x + u_x\xi_u)u_x + (\tau_x + u_x\tau_u)u_t$$

$$\eta^0_\alpha = \eta^\alpha_t + (\eta_u - \alpha(\tau_t + u_t\tau_u))u^\alpha_t - u(\eta_u)^\alpha_t - \sum_{n=1}^{\infty}\binom{\alpha}{n}D^n_t(\xi)D^{\alpha-n}_t(u_x)+$$
$$\sum_{n=1}^{\infty}[\binom{\alpha}{n}\partial^n_t\eta_u - \binom{\alpha}{n+1}D^{n+1}_t(\tau)]D^{\alpha-n}_t(u) + \mu,$$

where $\mu = \sum\limits_{n=2}^{\infty}\sum\limits_{m=2}^{n}\sum\limits_{k=2}^{m}\sum\limits_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}$, and

$$\eta^{xx} = \eta_{xx} + u_x\eta_{ux} - (\xi_{xx} + u_x\xi_{ux})u_x + (\tau_{xx} + u_x\tau_{ux})u_t - 2(\tau_x + u_x\tau_u)u_{xt}+$$
$$[\eta_{xu} + u_x\eta_{uu} - (\xi_{xu} + u_x\xi_{uu})u_x - (\tau_{xu} + u_x\tau_{uu})u_t]u_x+$$
$$u_{xx}(\eta_u - u_x\xi_u - u_t\tau_u) - 2(\xi_x + u_x\xi_u)u_{xx}$$

Using the expressions for η^x, η^0_α and η^{xx} in equation (3.2.6), we eventually arrive at the following:

$$\eta^\alpha_t + (\eta_u - \alpha(\tau_t + u_t\tau_u))Au_{xx} + B(u_x)^2 - u(\eta_u)^\alpha_t - \sum_{n=1}^{\infty}\binom{\alpha}{n}D^n_t(\xi)D^{\alpha-n}_t(u_x)+$$
$$\sum_{n=1}^{\infty}[\binom{\alpha}{n}\partial^n_t\eta_u - \binom{\alpha}{n+1}D^{n+1}_t(\tau)]D^{\alpha-n}_t(u) + \mu-$$
$$2Bu_x(\eta_x + u_x\eta_u - (\xi_x + u_x\xi_u)u_x + (\tau_x + u_x\tau_u)u_t)-$$
$$A(\eta_{xx} + u_x\eta_{ux} - (\xi_{xx} + u_x\xi_{ux})u_x + (\tau_{xx} + u_x\tau_{ux})u_t - 2(\tau_x + u_x\tau_u)u_{xt})+$$
$$A[\eta_{xu} + u_x\eta_{uu} - (\xi_{xu} + u_x\xi_{uu})u_x - (\tau_{xu} + u_x\tau_{uu})u_t]u_x+$$
$$Au_{xx}(\eta_u - u_x\xi_u - u_t\tau_u) - 2(\xi_x + u_x\xi_u)u_{xx} = 0. \quad \text{(3A-1)}$$

On equating the coefficients of different differentials equal to zero, we obtained a set of determining equations as (3.2.7)–(3.2.13)

Chapter 4

Similarity Reduction and Exact Solutions of a Variable Coefficient Space-Time Fractional Potential Burgers' Equation

4.1 Introduction

As mentioned in chapter 1, herein, a space-time fractional potential Burgers' (FPB) equation with variable coefficients

$$u_t^{(\alpha)} = f(t)u_x^{(2\beta)} + g(t)\left(u_x^{(\beta)}\right)^2, \quad x \in (0, \infty),\ t > 0,\ 0 < \alpha, \beta < 1, \tag{4.1.1}$$

where $u_t^{(\alpha)}$, $u_x^{(\beta)}$ are the modified Riemann-Liouville derivatives with respect to time and space variables respectively and the coefficients $f(t)$ and $g(t)$ are arbitrary smooth functions of variable t only, is examined for various exact solutions. The equation (4.1.1) is connected to the fractional Burgers' equation by the well-known Hopf-Cole transformation and is a generalisation of the time-fractional Burgers' equation with constant coefficients examined by Wu [104].

The chapter has been organized as follows. In Section 4.2, the symmetries for the FPB equation (4.1.1) are obtained. In Section 4.3, we analyze the reduced systems and find some invariant solutions of equation (4.1.1). Section 4.4 contains application of invariant subspace method on FPB equation (4.1.1). The final section contains the conclusion.

4.2 Symmetry Classification of FPB Equation

Herein, we investigate the symmetries of FPB equation (4.1.1). A fractional Lie symmetry of equation (4.1.1) is a continuous group of point transformations of independent and dependent variables which leaves the equation (4.1.1) invariant.

Let us assume that equation (4.1.1) admits the Lie symmetries of the form

$$\frac{\tilde{x^{\beta}}}{\Gamma(1+\beta)} = \frac{x^{\beta}}{\Gamma(1+\beta)} + \epsilon\xi(x,t,u) + o(\epsilon^2) \tag{4.2.1}$$

$$\frac{\tilde{t^{\alpha}}}{\Gamma(1+\alpha)} = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \epsilon\tau(x,t,u) + o(\epsilon^2) \tag{4.2.2}$$

$$\tilde{u} = u + \epsilon\eta(x,t,u) + o(\epsilon^2) \tag{4.2.3}$$

where ϵ is the group parameter and ξ, τ and η are the infinitesimals of the transformations for the independent and dependent variables, respectively. A group invariant solution of FPB equation (4.1.1) is a solution which can be mapped into another solution of equation (4.1.1) under the point transformations (4.2.1)–(4.2.3). The associated Lie algebra of infinitesimal symmetries of equation (4.1.1) is then the fractional vector field of the form

$$V = \xi(x,t,u)\frac{\partial^{\beta}}{\partial x^{\beta}} + \tau(x,t,u)\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{4.2.4}$$

The fractional second order prolongation of (4.2.4) is given by

$$\begin{aligned} pr^{(2)}V = \xi(x,t,u)\frac{\partial^{\beta}}{\partial x^{\beta}} + \tau(x,t,u)\frac{\partial^{\alpha}}{\partial t^{\alpha}} + \eta(x,t,u)\frac{\partial}{\partial u} + \eta^{t}\frac{\partial}{\partial u_t^{(\alpha)}} + \eta^{x}\frac{\partial}{\partial u_x^{(\beta)}} + \\ \eta^{tt}\frac{\partial}{\partial u_t^{(2\alpha)}} + \eta^{xt}\frac{\partial}{\partial (u_t^{(\alpha)})_x^{(\beta)}} + \eta^{xx}\frac{\partial}{\partial u_x^{(2\beta)}} \end{aligned} \tag{4.2.5}$$

Now for the invariance of equation (4.1.1) under equations (4.2.1–4.2.3), we must have

$$pr^{(2)}V([\Delta u])|_{([\Delta u])=0} = 0, \tag{4.2.6}$$

where $[\Delta u] = u_t^{(\alpha)} - f(t)u_x^{(2\beta)} - g(t)(u_x^{(\beta)})^2$ or equivalently, if

$$(\eta^t - f(t)\eta^{xx} - 2g(t)u_x^{(\beta)}\eta^x - \tau f_t^{(\alpha)}u_x^{(2\beta)} - \tau g_t^{(\alpha)}(u_x^{(\beta)})^2)|_{([\Delta u])=0} = 0. \tag{4.2.7}$$

The generalised fractional prolongation vector fields η^x, η^t and η^{xx} are given by

$$\eta^x = \eta_x^{(\beta)} + u_x^{(\beta)}\eta_u - (\xi_x^{(\beta)} + u_x^{(\beta)}\xi_u)u_x^{(\beta)} + (\tau_x^{(\beta)} + u_x^{(\beta)}\tau_u)u_t^{(\alpha)}$$

$$\eta^t = \eta_t^{(\alpha)} + u_t^{(\alpha)}\eta_u - (\xi_t^{(\alpha)} + u_t^{(\alpha)}\xi_u)u_x^{(\beta)} - (\tau_t^{(\alpha)} + u_t^{(\alpha)}\tau_u)u_t^{(\alpha)}$$

and

$$\begin{aligned}\eta^{xx} = \eta_x^{(2\beta)} + u_x^{(\beta)}(\eta_u)_x^{(\beta)} - (\xi_x^{(2\beta)} + u_x^{(\beta)}(\xi_u)_x^{(\beta)})u_x^{(\beta)} + (\tau_x^{(2\beta)} + u_x^{(\beta)}(\tau_u)_x^{(\beta)})u_t^{(\alpha)} - \\ 2(\tau_x^{(\beta)} + u_x^{(\beta)}\tau_u)(u_x^{(\beta)})_t^{(\alpha)} + u_x^{(2\beta)}(\eta_u - u_x^{(\beta)}\xi_u - u_t^{(\alpha)}\tau_u) - 2(\xi_x^{(\beta)} + u_x^{(\beta)}\xi_u)u_x^{(2\beta)} + \\ [(\eta_u)_x^{(\beta)} + u_x^{(\beta)}\eta_{uu} - ((\xi_u)_x^{(\beta)} + u_x^{(\beta)}\xi_{uu})u_x^{(\beta)} - ((\tau_u)_x^{(\beta)} + u_x^{(\beta)}\tau_{uu})u_t^{(\alpha)}]u_x^{(\beta)}\end{aligned}$$

Using the above generalised fractional prolongation vector fields in equation (4.2.7) and equating the coefficient of various derivative terms to zero, we get the following simplified set of determining equations (for details, refer to Appendix-4A)

$$\tau_u = 0 \tag{4.2.8}$$

$$\tau_x^{(\beta)} = 0 \tag{4.2.9}$$

$$\xi_u = 0 \tag{4.2.10}$$

$$2g(t)\xi_x^{(\beta)} - g(t)\tau_t^{(\alpha)} - \tau g_t^{(\alpha)} - g(t)\eta_u - f(t)\eta_{uu} = 0 \tag{4.2.11}$$

$$2f(t)\xi_x^{(\beta)} - f(t)\tau_t^{(\alpha)} - \tau f_t^{(\alpha)} = 0 \tag{4.2.12}$$

$$f(t)_0D_x^{2\beta}\xi - {}_0D_t^{\alpha}\xi - 2g(t)_0D_x^{\beta}\eta - 2f(t)_0D_x^{\beta}\eta_u = 0 \tag{4.2.13}$$

$$\eta_t^{(\alpha)} - f(t)\eta_x^{(2\beta)} = 0 \tag{4.2.14}$$

On solving equation (4.2.14) by using fractional Lie group method, a particular solution is obtained as

$$\eta = a_1 \frac{x^\beta}{\Gamma(1+\beta)} + a_2 F_t^{(\alpha)}(t) + a_2 \frac{x^{2\beta}}{\Gamma(1+2\beta)} + a_3$$

where $F_t^{(2\alpha)} = f_t^{(\alpha)}$ and a_1, a_2, a_3 are arbitrary constants. After substituting this value of η in the determining equations (4.2.11–4.2.12), we get the following infinitesimals

$$\xi = -2a_1 G(t) - 2a_2 G(t) \frac{x^\beta}{\Gamma(1+\beta)} + a_4 \frac{x^\beta}{\Gamma(1+\beta)} + a_5, \tag{4.2.15}$$

$$\tau = \frac{1}{F_t^{(2\alpha)}(t)} - [4a_2 H(t)_+ 2a_4 F_t^{(\alpha)}(t) + a_6], \tag{4.2.16}$$

$$\eta = a_1 \frac{x^\beta}{\Gamma(1+\beta)} + a_2 F_t^{(\alpha)}(t) + a_2 \frac{x^{2\beta}}{\Gamma(1+2\beta)} + a_3, \tag{4.2.17}$$

where $G_t^{(\alpha)}(t) = g(t)$, $H_t^{(\alpha)}(t) = F_t^{(2\alpha)}(t)G(t)$, and $a_1, a_2, ..., a_6$ are six arbitrary constants. Using equations (4.2.15–4.2.17) in equation (4.2.11) we also get $g(t) = kf(t)$, where k is an arbitrary constant. Further for $f(t) = g(t) = 1$ and $\beta = 1$ the infinitesimals can be reduced to those reported by Wu (2011), by setting the coefficients $a_1 = -c_5$, $a_2 = -2c_6$, $a_3 = c_3$, $a_4 = c_4$, $a_5 = c_1$, $a_6 = c_2$.
Hence, the fractional point symmetry generators admitted by the equation (4.1.1) are given by

$$V_1 = -2G(t) \frac{\partial^\beta}{\partial x^\beta} + \frac{x^\beta}{\Gamma(1+\beta)} \frac{\partial}{\partial u}$$

$$V_2 = -2G(t) \frac{x^\beta}{\Gamma(1+\beta)} \frac{\partial^\beta}{\partial x^\beta} - \frac{4H(t)}{F_t^{(2\alpha)}(t)} \frac{\partial^\alpha}{\partial t^\alpha} + [F_t^{(\alpha)}(t) + \frac{x^{2\beta}}{\Gamma(1+2\beta)} \frac{\partial}{\partial u}]$$

$$V_3 = \frac{\partial}{\partial u}$$

$$V_4 = \frac{x^\beta}{\Gamma(1+\beta)} \frac{\partial^\beta}{\partial x^\beta} + \frac{2F_t^{(\alpha)}(t)}{F_t^{(2\alpha)}(t)} \frac{\partial^\alpha}{\partial t^\alpha}$$

$$V_5 = \frac{\partial^\beta}{\partial x^\beta},$$

and

$$V_6 = \frac{1}{F_t^{(2\alpha)}(t)} \frac{\partial^\alpha}{\partial t^\alpha}$$

These infinitesimal generators can be used to determine a six parameter fractional Lie group of point transformation acting on (x, t, u)-space. It can be verified easily that the set $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ forms a six dimensional Lie algebra under the Lie bracket $[X, Y] = XY - YX$, which reduces to the well-known generalized Galilea algebra [84] for $\alpha = \beta = 1$. The commutator table is as given below:

Table 4.1: Commutator Table

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	0	0	V_1	$-V_3$	$2V_5$
V_2	0	0	0	$2V_2$	$2V_1$	$4V_4-2V_3$
V_3	0	0	0	0	0	0
V_4	$-V_1$	$-2V_2$	0	0	$-V_5$	$2V_6$
V_5	V_3	$-2V_1$	0	V_5	0	0
V_6	$-2V_5$	$2V_3-4V_4$	0	$-2V_6$	0	0

Further, from the commutator table it can be seen that the sets $\{V_3\}$ and $\{V_1, V_2, V_3\}$ form solvable subalgebras. Also, V_3 is the centre of the six dimensional Lie algebra as it commutes with every element of the Lie algebra. The group transformation generated by the infinitesimal generators $V_i, i = 1, 2, ..., 6$ is obtained by solving the system of ordinary differential equations

$$\frac{(d\tilde{x})^{\beta}}{\Gamma(1+\beta)d\epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) \tag{4.2.18}$$

$$\frac{(d\tilde{t})^{\alpha}}{\Gamma(1+\alpha)d\epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) \tag{4.2.19}$$

$$\frac{d\tilde{u}}{d\epsilon} = \eta(\tilde{x}, \tilde{t}, \tilde{u}), \tag{4.2.20}$$

with the initial conditions

$$\tilde{x}|_{\epsilon=0} = x \tag{4.2.21}$$

$$\tilde{t}|_{\epsilon=0} = t \tag{4.2.22}$$

$$\tilde{u}|_{\epsilon=0} = u \tag{4.2.23}$$

Exponentiating the infinitesimal symmetries of equation (4.1.1), we get the one parameter groups $g_i(\epsilon)$ generated by $V_i, i = 1, 2, ..., 6$

$$g_1 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)} - \frac{2\epsilon G}{\Gamma(1+\alpha)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u + \frac{\epsilon x^\beta}{\Gamma(1+\beta)}\right) \tag{4.2.24}$$

$$g_2 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)} e^{-2\epsilon G}, \frac{G}{(1+2\epsilon G)}, u + \frac{\epsilon F_t^{(\alpha)}}{(1+2\epsilon G)} + \frac{\epsilon x^{2\beta}}{\Gamma(1+2\beta)} e^{-\frac{4\epsilon G}{(1+2\epsilon G)}}\right) \tag{4.2.25}$$

$$g_3 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u + \epsilon\right) \tag{4.2.26}$$

$$g_4 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{e^\epsilon x^\beta}{\Gamma(1+\beta)}, \frac{e^{2\epsilon} t^\alpha}{\Gamma(1+\alpha)}, u\right) \tag{4.2.27}$$

$$g_5 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)} + \epsilon, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \tag{4.2.28}$$

$$g_6 : \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}, u\right) \to \left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)} + \epsilon, u\right) \tag{4.2.29}$$

Now, since g_i is a symmetry, if $u = \chi(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)})$ is a solution of equation (4.1.1) the following u_i are also solutions of equation (4.1.1)

$$u_1 = \chi\left(\frac{x^\beta}{\Gamma(1+\beta)} - \frac{2\epsilon G}{\Gamma(1+\alpha)}, \frac{t^\alpha}{\Gamma(1+\alpha)}\right) - \frac{\epsilon x^\beta}{\Gamma(1+\beta)} \tag{4.2.30}$$

$$u_2 = \chi\left(\frac{x^\beta}{\Gamma(1+\beta)} e^{-2\epsilon G}, \frac{G}{(1+2\epsilon G)}\right) - \frac{\epsilon F_t^{(\alpha)}}{(1+2\epsilon G)} - \frac{\epsilon x^{2\beta}}{\Gamma(1+2\beta)} e^{-\frac{4\epsilon G}{(1+2\epsilon G)}} \tag{4.2.31}$$

$$u_3 = \chi\left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)}\right) - \epsilon \tag{4.2.32}$$

$$u_4 = \chi\left(\frac{e^\epsilon x^\beta}{\Gamma(1+\beta)}, \frac{e^{2\epsilon} t^\alpha}{\Gamma(1+\alpha)}\right) \tag{4.2.33}$$

$$u_5 = \chi\left(\frac{x^\beta}{\Gamma(1+\beta)} + \epsilon, \frac{t^\alpha}{\Gamma(1+\alpha)}\right) \tag{4.2.34}$$

$$u_6 = \chi\left(\frac{x^\beta}{\Gamma(1+\beta)}, \frac{t^\alpha}{\Gamma(1+\alpha)} + \epsilon\right) \tag{4.2.35}$$

4.3 Some Exact Solutions of the FPB Equation

In this section, we investigate some exact solutions of equation (4.1.1) corresponding to following infinitesimal generators.

(i) V_1
(ii) V_4
(iii) $nV_5 + mV_3$
(iv) $rV_5 + V_6$ and
(v) $sV_3 + V_6$,
where r, s, m and n are non-zero arbitrary constant parameters.

Theorem 4.3.1 *Under the group of transformations* $T(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$ *and* $\phi(T) = \frac{x^{2\beta}}{\Gamma(1+2\beta)} + 2G(t)u(x,t)$*, the FPB equation (4.1.1) reduces to a linear differential equation of first order* $\phi'(T) - H_1(T)\phi(T) = -H_2(T)$*, where* $H_1(T) = \frac{G_t^{(\alpha)}(t)}{G(t)}$ *and* $H_2(T) = F_t^{(2\alpha)}$*. Which admits a solution given by*
$u(x,t) = \frac{1}{2G(t)}[e^{\int H_1 dT}\left(k_1 - \int H_2 e^{-\int H_1 dT} dT\right) - \frac{x^{2\beta}}{\Gamma(1+2\beta)}]$*, where* k_1 *is an arbitrary constant.*

Proof: Consider the infinitesimal generator V_1, given by

$$V_1 = -2G(t)\frac{\partial^\beta}{\partial x^\beta} + \frac{x^\beta}{\Gamma(1+\beta)}\frac{\partial}{\partial u} \tag{4.3.1}$$

We find an invariant solution of equation (4.1.1) by reducing it to a linear ordinary differential equation (4.3.4) using differential invariants.

The fractional characteristic equations for V_1 are

$$\frac{\frac{(dx)^\beta}{\Gamma(1+\beta)}}{-2G(t)} = \frac{\frac{(dt)^\alpha}{\Gamma(1+\alpha)}}{0} = \frac{du}{\frac{x^\beta}{\Gamma(1+\beta)}}. \tag{4.3.2}$$

On solving the above fractional characteristic equations we obtain two functionally independent invariants as $T = \frac{t^\alpha}{\Gamma(1+\alpha)}$, and $\nu = \frac{x^{2\beta}}{\Gamma(1+2\beta)} + 2G(t)u$.

Now the solution of the fractional characteristic equations will be of the form $\nu = \phi(T)$, therefore

$$u = \frac{1}{2G(T)}\left[\phi(T) - \frac{x^{2\beta}}{\Gamma(1+2\beta)}\right]. \tag{4.3.3}$$

Substituting this value of u in equation (4.1.1), we get the reduced linear ordinary differential equation as

$$\phi'(T) - H_1(T)\phi(T) = -H_2(T), \tag{4.3.4}$$

where $T = \frac{t^\alpha}{\Gamma(1+\alpha)}$, $H_1(T) = \frac{G_t^{(\alpha)}(t)}{G(t)}$ and $H_2(T) = F_t^{(2\alpha)}$. On solving equation (4.3.4), we obtain $\phi(T) = e^{\int H_1 dT}\left(k_1 - \int H_2 e^{-\int H_1 dT} dT\right)$, where k_1 is an arbitrary constant. This gives

$$u(x,t) = \frac{1}{2G(t)}\left[e^{\int H_1 dT}\left(k_1 - \int H_2 e^{-\int H_1 dT} dT\right) - \frac{x^{2\beta}}{\Gamma(1+2\beta)}\right] \tag{4.3.5}$$

Theorem 4.3.2 *The similarity transformations $u(x,t) = \psi(X)$ along with the similarity variable $X(x,t) = \frac{x^{2\beta}}{(\Gamma(1+\beta))^2 F_t^{(\alpha)}}$, reduces the FPB equation (4.1.1) to a nonlinear ordinary differential equation $\psi''(X) + \frac{g(t)}{f(t)}(\psi'(X))^2 + \frac{1}{4}\psi'(X) + \frac{1}{2X}\psi'(X) = 0$, which leads to the solution*

$u(x,t) = \frac{1}{k}\log\left\{k_2 + 2\sqrt{\pi}k \operatorname{erf}\left(\frac{x^\beta}{2\Gamma(1+\beta)\sqrt{F_t^{(\alpha)}}}\right)\right\} + k_3,$

where k_2 and k_3 are arbitrary constants.

Proof: Let us consider the infinitesimal generator

$$V_4 = \frac{x^\beta}{\Gamma(1+\beta)}\frac{\partial^\beta}{\partial x^\beta} + \frac{2F_t^{(\alpha)}(t)}{F_t^{(2\alpha)}(t)}\frac{\partial^\alpha}{\partial t^\alpha}. \tag{4.3.6}$$

Here the fractional characteristic equations give the invariants $X(x,t) = \frac{x^{2\beta}}{(\Gamma(1+\beta))^2 F_t^{(\alpha)}}$ and $u(x,t) = \psi(X)$. On substituting the above invariants in equation (4.1.1), it becomes a nonlinear ordinary differential equation of second order

$$\psi''(X) + \frac{g(t)}{f(t)}(\psi'(X))^2 + \frac{1}{4}\psi'(X) + \frac{1}{2X}\psi'(X) = 0, \tag{4.3.7}$$

which is further solved with the help of maple software to obtain the solution as

$$u(x,t) = \frac{1}{k}\log\left\{k_2 + 2\sqrt{\pi}k \operatorname{erf}\left(\frac{x^\beta}{2\Gamma(1+\beta)\sqrt{F_t^{(\alpha)}}}\right)\right\} + k_3. \tag{4.3.8}$$

Theorem 4.3.3 *Under the transformations $\zeta(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$ and $\varphi(\zeta) = \frac{x^\beta}{\Gamma(1+\beta)} - \frac{n}{m}u(x,t)$, the FPB equation (4.1.1) reduces to an ordinary fractional differential equation $\varphi_t^{(\alpha)} + \frac{m}{n}G_t^{(\alpha)} = 0$, with $G_t^{(\alpha)} = g(t)$. This has the general solution as $u(x,t) = \frac{m}{n}\left[\frac{x^\beta}{\Gamma(1+\beta)} + \frac{m}{n}G(t) - k_4\right]$, with k_4 as an arbitrary constant.*

Proof: In this case, we study the infinitesimal generator

$$V = nV_5 + mV_3 = n\frac{\partial^\beta}{\partial x^\beta} + m\frac{\partial}{\partial u} \tag{4.3.9}$$

The following invariants can be derived easily $\zeta(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$ and $\nu = \frac{x^\beta}{\Gamma(1+\beta)} - \frac{n}{m}u(x,t)$ and the reduced form of eqn. (4.1.1) as

$$\varphi_t^{(\alpha)} + \frac{m}{n}G_t^{(\alpha)} = 0. \tag{4.3.10}$$

This easily yields the solution

$$u(x,t) = \frac{m}{n}\left[\frac{x^\beta}{\Gamma(1+\beta)} + \frac{m}{n}G(t) - k_4\right]. \tag{4.3.11}$$

Theorem 4.3.4 *Under the transformations $\mu(x,t) = \frac{1}{r}\frac{x^\beta}{\Gamma(1+\beta)} - \frac{F_t^{(\alpha)}}{\Gamma(1+\alpha)}$ and $\omega(\mu) = u(x,t)$, the the FPB equation (4.1.1) reduces to a nonlinear ordinary differential equation $\omega''(\mu) + k\omega'^2(\mu) + \frac{r^2}{\Gamma(1+\alpha)}\omega'(\mu) = 0$, which admits the solution*

$$u(x,t) = \frac{1}{k}\log\left\{e^{\frac{r^2}{\Gamma(1+\alpha)}\left\{\frac{x^\beta}{r\Gamma(1+\beta)} - \frac{F_t^{(\alpha)}}{\Gamma(1+\alpha)}\right\}} - ke^{\frac{c_1 r^2}{\Gamma(1+\alpha)}}\right\} - \frac{r^2}{\Gamma(1+\alpha)}\left\{\frac{x^\beta}{r\Gamma(1+\beta)} - \frac{F_t^{(\alpha)}}{\Gamma(1+\alpha)}\right\} + c_2,$$

where c_1 and c_2 are arbitrary constants.

Proof: Results can be easily derived by solving the fractional characteristic equations for the infinitesimal generator $rV_5 + V_6$.

Theorem 4.3.5 *The similarity transformations $\gamma(x,t) = \frac{x^\beta}{\Gamma(1+\beta)}$ and $\rho(\gamma) = -\frac{\Gamma(1+\alpha)}{s}u + F_t^{(\alpha)}$, reduce the FPB equation (4.1.1) to a nonlinear ordinary differential equation $\rho''(\gamma) - \frac{ks}{\Gamma(1+\alpha)}(\rho'(\gamma))^2 + 1 = 0$. Which has the general solution as $u(x,t) = \frac{s}{\Gamma(1+\alpha)}\left[F_t^{(\alpha)} + \frac{1}{k}\log\left\{\cosh\sqrt{k}(c_3 + \frac{x^\beta}{\Gamma(1+\beta)})\right\} - c_4\right]$, where c_3 and c_4 are arbitrary constants.*

Proof: The proof is similar to the previous theorems.

4.4 Some Exact Solutions of FPB Equation by the Invariant Subspace Method

The invariant subspace method is one of the recently developed techniques to construct an exact solution of nonlinear partial differential equations of evolution type. In this section, we utilize the method, as described in section 1.6 of chapter 1, to discover some exact solutions of equation (4.1.1).

For equation (4.1.1), $F[u] = f(t)u_x^{2\beta} + g(t)(u_x^\beta)^2$. We have the space $W_3 = \langle 1, \frac{x^\beta}{\Gamma(1+\beta)}, \frac{x^{2\beta}}{\Gamma(1+2\beta)} \rangle$ as invariant under $F[u]$, if and only if, for any $u(x,t) = a_1(t) + a_2(t)\frac{x^\beta}{\Gamma(1+\beta)} + a_3(t)\frac{x^{2\beta}}{\Gamma(1+2\beta)} \in W_3$, $F[u] \in W_3$
or equivalently iff,

$$a_3(t)f(t) + (a_2(t))^2 g(t) = b_1$$
$$(a_3(t))^2 g(t) = b_3, \text{ and}$$
$$a_2(t)a_3(t)g(t) = b_2,$$

where b_1,b_2 and b_3 are arbitrary constants.

This allows us to consider an exact solution of equation (4.1.1) as

$$u(x,t) = a_1(t) + a_2(t)\frac{x^\beta}{\Gamma(1+\beta)} + a_3(t)\frac{x^{2\beta}}{\Gamma(1+2\beta)}. \tag{4.4.1}$$

Substituting the value of $u(x,t)$ from equation (4.4.1) into the equation (4.1.1) and equating the coefficients of the elements of space $W_3 = \langle 1, \frac{x^\beta}{\Gamma(1+\beta)}, \frac{x^{2\beta}}{\Gamma(1+2\beta)} \rangle$, we get the following system of fractional differential equations

$$\frac{d^\alpha a_3(t)}{dt^\alpha} = g(t)(a_3(t))^2 \tag{4.4.2}$$

$$\frac{d^\alpha a_2(t)}{dt^\alpha} = a_2(t)a_3(t)g(t) \tag{4.4.3}$$

and

$$\frac{d^\alpha a_1(t)}{dt^\alpha} = f(t)a_3(t) + g(t)(a_2(t))^2. \tag{4.4.4}$$

Equations (4.4.2–4.4.4) can be readily solved to yield

$$-\frac{1}{a_3(t)} = \frac{\Gamma(1+2\beta)}{(\Gamma(1+\beta))^2}({}_0I_t^\alpha g) \tag{4.4.5}$$

$$\log a_2(t) = 2({}_0I_t^\alpha(a_3(t)g)) \tag{4.4.6}$$

and

$$a_1(t) = {}_0I_t^\alpha(a_3(t)f + (a_2(t))^2 g), \tag{4.4.7}$$

where the functions $f(t)$ and $g(t)$ are integrable in the sense of Riemann-Liouville and ${}_0I_t^\alpha g(t) \neq 0$.

On solving equations (4.4.5)–(4.4.7), we get

$$a_3(t) = \frac{B}{{}_0I_t^\alpha g(t)} \tag{4.4.8}$$

$$a_2(t) = ({}_0I_t^\alpha g(t))^{2B} \tag{4.4.9}$$

and

$$a_1(t) = {}_0I_t^\alpha[g(t)({}_0I_t^\alpha g(t))^{4B} + \frac{Bf(t)}{{}_0I_t^\alpha g(t)}], \tag{4.4.10}$$

where $B = -\frac{(\Gamma(1+\beta))^2}{\Gamma(1+2\beta)}$.

Using equation (4.4.1) and equations (4.4.8)–(4.4.10) one can easily obtain an exact solution of equation (4.1.1).

4.5 Discussion

The main purpose of Lie symmetry method is to reduce PDEs to ODEs by introducing suitable similarity variable and similarity solutions. Here, in this chapter, It has been shown that the FPB equation possess similarity solutions, exactly as its counterparts with integer-order derivatives. By using conveniently defined similarity variables the FPB equation reduces to ordinary differential equations which are solved to derive some group invariant solutions. Further, the invariant subspace method has been utilised to deduce some exact solutions of the FPB equation. The software like Mathematica and Maple have been utilised in solving some ordinary differential equations.

Appendix–4A

The extended infinitesimals η^t, η^x and η^{xx} can be easily obtained as

$$\eta^t = \eta_t^{(\alpha)} + u_t^{(\alpha)}\eta_u - (\xi_t^{(\alpha)} + u_t^{(\alpha)}\xi_u)u_x^{(\beta)} - (\tau_t^{(\alpha)} + u_t^{(\alpha)}\tau_u)u_t^{(\alpha)},$$

$$\eta^x = \eta_x^{(\beta)} + u_x^{(\beta)}\eta_u - (\xi_x^{(\beta)} + u_x^{(\beta)}\xi_u)u_x^{(\beta)} + (\tau_x^{(\beta)} + u_x^{(\beta)}\tau_u)u_t^{(\alpha)},$$

$$\begin{aligned}\eta^{xx} = {} & \eta_x^{(2\beta)} + u_x^{(\beta)}(\eta_u)_x^{(\beta)} - (\xi_x^{(2\beta)} + u_x^{(\beta)}(\xi_u)_x^{(\beta)})u_x^{(\beta)} + (\tau_x^{(2\beta)} + u_x^{(\beta)}(\tau_u)_x^{(\beta)})u_t^{(\alpha)} - \\ & 2(\tau_x^{(\beta)} + u_x^{(\beta)}\tau_u)(u_x^{(\beta)})_t^{(\alpha)} + u_x^{(2\beta)}(\eta_u - u_x^{(\beta)}\xi_u - u_t^{(\alpha)}\tau_u) - 2(\xi_x^{(\beta)} + u_x^{(\beta)}\xi_u)u_x^{(2\beta)} + \\ & [(\eta_u)_x^{(\beta)} + u_x^{(\beta)}\eta_{uu} - ((\xi_u)_x^{(\beta)} + u_x^{(\beta)}\xi_{uu})u_x^{(\beta)} - ((\tau_u)_x^{(\beta)} + u_x^{(\beta)}\tau_{uu})u_t^{(\alpha)}]u_x^{(\beta)},\end{aligned}$$

Using the expressions for η^t, η^x and η^{xx} in equation (4.2.7), we eventually arrive at the following:

$$\begin{aligned}& \eta(u_x^{(\beta)} - u_x^{(3\beta)}) + (1 + u - 3u_x^{(2\beta)})(\eta_x^{(\beta)} + u_x^{(\beta)}\eta_u - (\xi_x^{(\beta)} + u_x^{(\beta)}\xi_u)u_x^{(\beta)} + (\tau_x^{(\beta)} + u_x^{(\beta)}\tau_u)u_t^{(\alpha)}) + \\ & \eta_t^{(\alpha)} + u_t^{(\alpha)}\eta_u - (\xi_t^{(\alpha)} + u_t^{(\alpha)}\xi_u)u_x^{(\beta)} - (\tau_t^{(\alpha)} + u_t^{(\alpha)}\tau_u)u_t^{(\alpha)} - \\ & 3u_x^{(\beta)}(\eta_x^{(2\beta)} + u_x^{(\beta)}(\eta_u)_x^{(\beta)} - (\xi_x^{(2\beta)} + u_x^{(\beta)}(\xi_u)_x^{(\beta)})u_x^{(\beta)} + (\tau_x^{(2\beta)} + u_x^{(\beta)}(\tau_u)_x^{(\beta)})u_t^{(\alpha)}) - \\ & 6u_x^{(\beta)}((\tau_x^{(\beta)} + u_x^{(\beta)}\tau_u)(u_x^{(\beta)})_t^{(\alpha)} + u_x^{(2\beta)}(\eta_u - u_x^{(\beta)}\xi_u - u_t^{(\alpha)}\tau_u) - 2(\xi_x^{(\beta)} + u_x^{(\beta)}\xi_u)u_x^{(2\beta)}) + \\ & 3u_x^{(\beta)}([(\eta_u)_x^{(\beta)} + u_x^{(\beta)}\eta_{uu} - ((\xi_u)_x^{(\beta)} + u_x^{(\beta)}\xi_{uu})u_x^{(\beta)} - ((\tau_u)_x^{(\beta)} + u_x^{(\beta)}\tau_{uu})u_t^{(\alpha)}]u_x^{(\beta)}) - \\ & u(\eta_x^{(3\beta)} + (3(\eta_u)_x^{(2\beta)} - \xi_x^{(3\beta)})u_x^{(\beta)} - \tau_x^{(3\beta)}u_t^{(\alpha)} - 3(\tau_u)_x^{(2\beta)}u_t^{(\alpha)}u_x^{(\beta)}) + \\ & u((3(\eta_{uu})_x^{(\beta)} - (\xi_u)_x^{(2\beta)})(u_x^{(\beta)})^2 - 3(\tau_{uu})_x^{(\beta)}u_t^{(\alpha)}(u_x^{(\beta)})^2 + ((\eta_{uuu}) - 3(\xi_{uu})_x^{(\beta)})(u_x^{(\beta)})^3) - \\ & u(\tau_{uuu}u_t^{(\alpha)}(u_x^{(\beta)})^3 - \xi_{uuu}(u_x^{(\beta)})^4 - 3\tau_{xx}(u_x^{(\beta)})_t^{(\alpha)} + (3(\eta_u)_x^{(\beta)} - \xi_x^{(2\beta)})u_x^{(2\beta)}) + \\ & u(3((\eta_{uu}) - 3(\xi_u)_x^{(\beta)})u_x^{(\beta)}u_x^{(2\beta)} - 3(\tau_u)_x^{(\beta)}u_t^{(\alpha)}u_x^{(2\beta)} - 6(\tau_u)_x^{(\beta)}u_x^{(\beta)}(u_x^{(\beta)})_t^{(\alpha)} - 6\xi_{uu}u_x^{(2\beta)}(u_x^{(\beta)})^2) - \\ & u(3\tau_{uu}(u_x^{(\beta)})^2(u_x^{(\beta)})_t^{(\alpha)} - 3\tau_{uu}u_x^{(2\beta)}u_x^{(\beta)}u_t^{(\alpha)} - 3\xi_u(u_x^{(2\beta)})^2 - 3\tau_u u_x^{(2\beta)}(u_x^{(\beta)})_t^{(\alpha)} - 3\tau_x(u_x^{(2\beta)})_t^{(\alpha)}) + \\ & u((\eta_u - 3\xi_x^{(\beta)})(u_x^{(3\beta)}) - 4\xi_u(u_x^{(3\beta)})u_x^{(\beta)} - \tau_u(u_x^{(3\beta)})u_t^{(\alpha)} - 3\tau_u(u_x^{(\beta)})(u_x^{(2\beta)})_t^{(\alpha)}) - \eta^{xxt}|_{([\Delta u])=0} = 0\end{aligned} \tag{4A-1}$$

On equating the coefficients of different differentials equal to zero, we obtained a set of determining equations as (4.2.8)–(4.2.14)

Chapter 5

Lie Group of Transformations of Time Fractional Gardner Equation

5.1 Introduction

During the past few decades the mathematical theory of the nonlinear evolution equations, such as the Korteweg-de Vries (KdV) equation, modified Korteweg-de Vries (KdV) equation, Boussinesq equation, Peregrine equation, and other such models that describe a large variety of physical phenomena has gained a lot of attention of many researchers [93, 100]. The Korteweg-de Vries equation (KdV) is a well-known model to describe nonlinear long internal waves in the ocean. Its coefficients are defined by vertical density and currents stratification. A detailed study on the KdV equation with variable coefficients was presented by Zhang [105] using the exponential function method. There are various situations in nature where it becomes necessary to consider quadratic nonlinearity. For example, in a density stratified ocean, where internal gravity waves are observed, the single nonlinear term does not correctly model the shallow water waves. In the Coastal Ocean Probe Experiment (COPE) conducted in Oregon Bay in 1995, strong nonlinearity was experienced in the internal gravity waves. Therefore, the problem of creating an adequate theoretical model was deemed necessary. The shallow water wave engineering experiments led to the construction of a wave equation with dual-power law nonlinearity. This leads to the study of Gardner equation [26], which is the simplest model that illustrates this effect. The Gardner equation is a modified version of the KdV equation and is also known as the mixed KDV-mKdV equation. It differs from the KdV equation by presence of an additional term of cubic nonlinearity. The Gardner equation shows up the internal gravity waves in a density-stratified ocean which is commonly described by the KdV equations and its versions with small nonlinearity.

In this chapter, we investigated the Lie symmetries of the time fractional Gardner equation of the form

$$u_t^{(\alpha)} = Auu_x + Bu^2u_x + u_{xxx}, \quad x \in (0, \infty),\ t > 0,\ 0 < \alpha < 1, \tag{5.1.1}$$

where A and B are real constant parameters.
The chapter has been organized as follows. In Section 5.2, Lie symmetries of the time fractional Gardner equation has been investigated. In Section 5.3, the reduced systems and some invariant solutions of the time fractional Gardner equation (5.1.1) are presented. Section 5.4 is devoted to the application of invariant subspace method on time fractional Gardner equation (5.1.1). The final section contains the conclusion.

5.2 Lie Symmetries of Time Fractional Gardner Equation

Herein, we investigate the symmetries of time fractional Gardner equation (5.1.1). A fractional Lie symmetry of equation (5.1.1) is a continuous group of point transformations of independent and dependent variables which leaves the equation (5.1.1) invariant.

Let us assume that equation (5.1.1) admits the Lie symmetries of the form

$$\tilde{x} = x + \epsilon\xi(x,t,u) + o(\epsilon^2) \tag{5.2.1}$$

$$\tilde{t} = t + \epsilon\tau(x,t,u) + o(\epsilon^2) \tag{5.2.2}$$

$$\tilde{u} = u + \epsilon\eta(x,t,u) + o(\epsilon^2), \tag{5.2.3}$$

where ϵ is the group parameter and ξ, τ and η are the infinitesimals of the transformations for the independent and dependent variables, respectively. A group invariant solution of time fractional Gardner equation(5.1.1) is a solution which can be mapped into another solution of equation (5.1.1) under the point transformations (5.2.1–5.2.3). The associated Lie algebra of infinitesimal symmetries of equation (5.1.1) is then the vector field of the form

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{5.2.4}$$

The fractional third order prolongation of (5.2.4) is

$$pr^{(3)}V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} + \eta^t\frac{\partial}{\partial u_t} + \eta^x\frac{\partial}{\partial u_x} + \eta_\alpha^0\frac{\partial}{\partial u_t^{(\alpha)}} + \eta^{ttt}\frac{\partial}{\partial u_{ttt}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}}. \tag{5.2.5}$$

Invoking the invariance criterion as explained in chapter 1, the following relation from the coefficients of the first order of ϵ is deduced:

$$\eta_\alpha^0 - A(\eta u_x + \eta^x u) - B(2\eta u u_x + \eta^x u^2) - \eta^{xxx}|_{[\Delta u]=0} = 0, \tag{5.2.6}$$

where $[\Delta u] = u_t^{(\alpha)} - Auu_x - Bu^2u_x - u_{xxx}$ and η^x, η_α^0 and η^{xxx} are extended infinitesimals acting on enlarged space corresponding to u_x, u_t^α and u_{xxx}. The general solution of equation (5.2.6) provides the infinitesimal elements ξ, τ and η for which the equation (5.1.1) possesses Lie symmetry. The generalised fractional prolongation vector fields, η^x, η_α^0 and η^{xxx} are given by

$$\eta^x = \eta_x + u_x\eta_u - (\xi_x + u_x\xi_u)u_x - (\tau_x + u_x\tau_u)u_t \tag{5.2.7}$$

$$\eta_\alpha^0 = \eta_t^\alpha + (\eta_u - \alpha(\tau_t + u_t\tau_u))u_t^\alpha - u(\eta_u)_t^\alpha - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi)D_t^{\alpha-n}(u_x)+$$
$$\sum_{n=1}^{\infty}[\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau)]D_t^{\alpha-n}(u) + \mu, \quad (5.2.8)$$

where $\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n-\alpha+1)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}$, and

$$\eta^{xxx} = \eta_{xxx} + (3\eta_{uxx} - \xi_{xxx})u_x - \tau_{xxx}u_t - 3\tau_{uxx}u_t u_x+$$
$$(3\eta_{uux} - \xi_{uxx})(u_x)^2 - 3\tau_{uux})u_t(u_x)^2 + (\eta_{uuu} - 3\xi_{uux})(u_x)^3-$$
$$\tau_{uuu}u_t(u_x)^3 - \xi_{uuu}(u_x)^4 - 3\tau_{xx}u_{xt} + (3\eta_{ux} - \xi_{xx})u_{xx}+$$
$$3(\eta_{uu} - 3\xi_{ux})u_x u_{xx} - 3\tau_{ux}u_t u_{xx} - 6\tau_{ux}u_x u_{xt} - 6\xi_{uu}u_{xx}(u_x)^2-$$
$$3\tau_{uu}(u_x)^2 u_{xt} - 3\tau_{uu}u_{xx}u_x u_t - 3\xi_u(u_{xx})^2 - 3\tau_u u_{xx}u_{xt} - 3\tau_x u_{xxt}+$$
$$(\eta_u - 3\xi_x)u_{xxx} - 4\xi_u u_{xxx}u_x - \tau_u u_{xxx}u_t - 3\tau_u u_x u_{xxt} \quad (5.2.9)$$

Using the expressions for η^x, η_α^0 and η^{xx} in equation (5.2.6) and equating the coefficient of various derivative terms to zero, we get the following simplified set of determining equations

$$\tau_x = \tau_u = 0 \quad (5.2.10)$$

$$\eta_{uu} = \eta_{ux} = 0 \quad (5.2.11)$$

$$\xi_u = 0 \quad (5.2.12)$$

$$\binom{\alpha}{n}\partial_t^n\eta - \binom{\alpha}{n+1}D_t^{n+1}(\tau) = 0, \quad n = 1, 2, 3, ... \quad (5.2.13)$$

$$(\eta_u - \alpha\tau_t)(Au + Bu^2) - (A + 2Bu)\eta - (Au + Bu^2)(\eta_u - \xi_x) = 0 \quad (5.2.14)$$

$$\eta_t^\alpha - u(\eta_u)_t^\alpha - (Au + Bu^2)\eta_x - \eta_{xxx} = 0 \quad (5.2.15)$$

$$D_t^n(\xi) = 0, \quad n = 1, 2, 3, ... \quad (5.2.16)$$

On solving the set of determining equations (5.2.10)–(5.2.16) obtained from the invariance condition for time fractional Gardner equation, we arrive at the following two cases (i) $A = 0$ or (ii) $B = 0$

Case (i) if $A = 0$,

In this case the infinitesimals are obtained as

$$\xi = c_1 x + c_2 \quad (5.2.17)$$

$$\tau = \frac{3c_1 t}{\alpha} \tag{5.2.18}$$

$$\eta = -c_1 u \tag{5.2.19}$$

where c_1, c_2 are arbitrary parameters. The point symmetry generators admitted by the time fractional Gardner equation are given by

$$V_1 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} \tag{5.2.20}$$

$$V_2 = \frac{\partial}{\partial x}. \tag{5.2.21}$$

Hence, the infinitesimal operator (5.2.4) becomes $V = (c_1 x + c_2)\frac{\partial}{\partial x} + \frac{3c_1 t}{\alpha}\frac{\partial}{\partial t} - c_1\frac{\partial}{\partial u}$. Further, these infinitesimal generators (5.2.20-5.2.21) can be used to determine a two parameters Lie group of point transformations acting on (x, t, u)-space which is fewer than those for the standard Gardner equation [26]. It can be verified easily that the set $\{V_1, V_2\}$forms a two dimensional Lie algebra under the Lie bracket $[X, Y] = XY - YX$ and its commutator table is as given below:

Table 5.1: Commutator Table

	V_1	V_2
V_1	0	-V_2
V_2	V_2	0

The group transformation generated by the infinitesimal generators $V_i, i = 1, 2$ is obtained by solving the system of ordinary differential equations

$$\frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) \tag{5.2.22}$$

$$\frac{d\tilde{t}}{d\epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) \tag{5.2.23}$$

$$\frac{d\tilde{u}}{d\epsilon} = \eta(\tilde{x}, \tilde{t}, \tilde{u}) \tag{5.2.24}$$

with the initial conditions

$$\tilde{x}|_{\epsilon=0} = x \tag{5.2.25}$$

$$\tilde{t}|_{\epsilon=0} = t \tag{5.2.26}$$

$$\tilde{u}|_{\epsilon=0} = u. \tag{5.2.27}$$

Exponentiating the infinitesimal symmetries of equation (5.1.1), we get the one parameter groups $g_i(\epsilon)$ generated by $V_i, i = 1, 2$

$$g_1 : (x, t, u) \to (e^{\epsilon}x, e^{\frac{3}{\alpha}}t, e^{-\epsilon}u) \tag{5.2.28}$$

$$g_2 : (x, t, u) \to (x + \epsilon, t, u) \tag{5.2.29}$$

Case (ii) if $B = 0$

In this case we obtain the infinitesimals as

$$\xi = c_1x + c_2 \tag{5.2.30}$$

$$\tau = \frac{3c_1t}{\alpha} \tag{5.2.31}$$

$$\eta = -2c_1u \tag{5.2.32}$$

where c_1, c_2 are arbitrary parameters.

The point symmetry generators obtained in this case are

$$V_3 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}, \tag{5.2.33}$$

$$V_4 = \frac{\partial}{\partial x}. \tag{5.2.34}$$

The infinitesimal operator (5.2.4) in this case becomes $V = (c_1x+c_2)\frac{\partial}{\partial x} + \frac{3c_1t}{\alpha}\frac{\partial}{\partial t} - 2c_1\frac{\partial}{\partial u}$, and the group transformation generated by the infinitesimal generators $V_i, i = 3, 4$ are

$$g_3 : (x, t, u) \to (e^{\epsilon}x, e^{\frac{3}{\alpha}}t, e^{-2\epsilon}u) \tag{5.2.35}$$

$$g_4 : (x, t, u) \to (x + \epsilon, t, u). \tag{5.2.36}$$

Further, the commutator table for the two dimensional Lie algebra constituted by $\{V_3, V_4\}$ is as given below:

Table 5.2: Commutator Table

	V_3	V_4
V_3	0	-V_4
V_4	V_4	0

5.3 Reduction to ODE

Herein, we reduce the time fractional Gardner equation (5.1.1) to ODE with the Erdélyi-Kober fractional differential operator.
Case (i) if $A = 0$

For the infinitesimal generator V_1 the characteristic equations are

$$\frac{dx}{x} = \frac{\alpha dt}{3t} = \frac{du}{-u}, \tag{5.3.1}$$

which give the invariants as $u(x,t) = x^{-1}f(z)$, $z = xt^{\frac{-\alpha}{3}}$. Corresponding to these invariants we can reduce equation (5.1.1) to an ODE of fractional order. We summarize the result in the following theorem :

Theorem 5.3.1 *The similarity transformation $u(x,t) = x^{-1}f(z)$ along with the similarity variable $z = xt^{\frac{-\alpha}{3}}$ reduces the time fractional Gardner equation to the ordinary differential equation of fractional order of the form*

$$(P_{\frac{3}{\alpha}}^{1-\frac{5}{3}\alpha,\alpha} f)(z) = \frac{d^3 f}{dz^3} + Bf^2 \frac{df}{dz} \tag{5.3.2}$$

with the Erdélyi-Kober fractional differential operator

$$(P_\delta^{\tau,\alpha} f)(z) = \prod_{j=0}^{m-1} (\tau + j - \frac{1}{\delta} z \frac{d}{dz})(K_\delta^{\tau+\alpha,m-\alpha} f)(z), \; z > 0, \; \delta > 0, \; \alpha > 0, \tag{5.3.3}$$

$$m = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N}. \end{cases}, \quad \textit{where}$$

$$(K_\delta^{\tau,\alpha} f)\,(z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_0^\infty (\nu-1)^{\alpha-1}\nu^{-(\tau+\alpha)} f(z\nu^{\frac{1}{\delta}})\,d\nu\,, & \alpha > 0; \\ f(z), & \alpha = 0 \end{cases} \tag{5.3.4}$$

is the Erdélyi-Kober fractional integral operator.

Proof: Let $n-1 < \alpha < n$, $n = 1,2,3,...$ then the Riemann-Liouville fractional derivative for the similarity transformation $u(x,t) = f(z)$ with the similarity variable $z = xt^{\frac{-\alpha}{3}}$ becomes

$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}[\frac{1}{\Gamma(n-\alpha)}\int\limits_0^t (t-s)^{n-\alpha-1} f(xs^{\frac{-\alpha}{3}})\,ds]$.

Let $\nu = \frac{t}{s}$. Then the above equation can be written as

$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}[t^{n-\alpha}\frac{1}{\Gamma(n-\alpha)}\int\limits_1^\infty (\nu-1)^{n-\alpha-1}\nu^{-(n-\alpha+1)} f(z\nu^{\frac{\alpha}{3}})\,d\nu]$.

Following the definition of the Erdélyi-Kober fractional integral operator given in equation (5.3.4), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}[t^{n-\alpha}(K_{\frac{3}{\alpha}}^{1,n-\alpha} f)(z)]. \tag{5.3.5}$$

In order to simplify the right hand side of equation (5.3.5), we consider the relation $z = xt^{\frac{-\alpha}{3}}$, $f \in C^1(0,\infty)$,

$$\begin{aligned} t\frac{\partial}{\partial t} f(z) &= tx(-\frac{\alpha}{3})t^{-\frac{\alpha}{3}-1} f'(z) \\ &= -\frac{\alpha}{3} z\frac{d}{dz} f(z) \end{aligned}$$

and thus, we get

$$\begin{aligned} \frac{\partial^n}{\partial t^n}[t^{n-\alpha}(K_{\frac{3}{\alpha}}^{1,n-\alpha} f)(z)] &= \frac{\partial^{n-1}}{\partial t^{n-1}}[\frac{\partial}{\partial t}(t^{n-\alpha}(K_{\frac{3}{\alpha}}^{1,n-\alpha} f)(z))] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}[t^{n-\alpha-1}(n-\alpha-\frac{\alpha}{3}z\frac{d}{dz}(K_{\frac{2}{\alpha}}^{1,n-\alpha} f)(z))]. \end{aligned}$$

Repeating the similar procedure for $n-1$ times, we have

$$\frac{\partial^n}{\partial t^n}[t^{n-\alpha}(K_{\frac{3}{\alpha}}^{1,n-\alpha} f)(z)] = t^{-\alpha}\prod_{j=0}^{n-1}(1-\alpha+j-\frac{\alpha}{3}z\frac{d}{dz})(K_{\frac{3}{\alpha}}^{1,n-\alpha} f)(z). \tag{5.3.6}$$

Now using the definition of Erdélyi-Kober fractional differential operator given in equation (5.3.3), the above equation can be written as

$$\frac{\partial^n}{\partial t^n}[t^{n-\alpha}(K_{\frac{3}{\alpha}}^{1,n-\alpha}f)(z)] = t^{-\alpha}(P_{\frac{3}{\alpha}}^{1-\frac{5}{3}\alpha,\alpha}f)(z). \tag{5.3.7}$$

We obtain an expression for the time fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha}(P_{\frac{3}{\alpha}}^{1-\frac{5}{3}\alpha,\alpha}f)(z). \tag{5.3.8}$$

Continuing further we find that the time fractional Gardner equation (5.1.1) reduces to an ordinary differential equation of fractional order

$$(P_{\frac{3}{\alpha}}^{1-\frac{5}{3}\alpha,\alpha}f)(z) = \frac{d^3f}{dz^3} + Bf^2\frac{df}{dz}. \tag{5.3.9}$$

Due to the arbitrary order $0 < \alpha < 1$ the reduced equation (5.3.9) is not solvable in general. However, for some special cases, such as the initial value problems and the linear equations, the solutions can be furnished by the power series method with Mittag-Leffler function and Wright and the generalised Wright functions. In particular, when $B = 0$, three independent solutions of equation (5.1.1) can be derived [18].

Case (ii) if $B = 0$

For the infinitesimal generator V_3 the characteristic equations are

$$\frac{dx}{x} = \frac{\alpha dt}{3t} = \frac{du}{-2u}, \tag{5.3.10}$$

which give the invariants as $u(x,t) = x^{-2}f(z)$, $z = xt^{\frac{-\alpha}{3}}$. Corresponding to these invariants the time fractional Gardner equation (5.1.1) is reduced to an ODE of fractional order. The result has been summarized in the following theorem :

Theorem 5.3.2 *The similarity transformation $u(x,t) = x^{-2}f(z)$ along with the similarity variable $z = xt^{\frac{-\alpha}{3}}$ reduces the time fractional Gardner equation to the ordinary differential equation of fractional order of the form*

$$(P_{\frac{3}{\alpha}}^{1-\frac{4}{3}\alpha,\alpha}f)(z) = \frac{d^3f}{dz^3} + Af\frac{df}{dz} \tag{5.3.11}$$

with the Erdélyi-Kober fractional differential operator

$$(P_\delta^{\tau,\alpha} f)(z) = \prod_{j=0}^{m-1}(\tau + j - \frac{1}{\delta} z \frac{d}{dz})(K_\delta^{\tau+\alpha, m-\alpha} f)(z), \; z > 0, \; \delta > 0, \; \alpha > 0, \tag{5.3.12}$$

$$m = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N}. \end{cases} , \quad \textit{where}$$

$$(K_\delta^{\tau,\alpha} f)\;(z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_0^\infty (\nu - 1)^{\alpha-1} \nu^{-(\tau+\alpha)} f(z\nu^{\frac{1}{\delta}})\, d\nu\,, & \alpha > 0; \\ f(z), & \alpha = 0 \end{cases} \tag{5.3.13}$$

is the Erdélyi-Kober fractional integral operator.

Proof: The proof is similar to Theorem 5.3.1.
Here again the nonlinear differential equation of fractional order, (5.3.11), is not solvable, in general. However, when $A = 0$ one can derive three independent solutions of it following the procedure given in [18].

5.4 Some Exact Solutions of Time Fractional Gardner Equation by the Invariant Subspace Method.

In this section the invariant subspace method (as defined in section 1.6) has been utilized in order to discover exact solutions of equation (5.1.1) for $B = 0$.
For equation (5.1.1),

$$F[u] = Auu_x + u_{xxx}. \tag{5.4.1}$$

We have the space $W_2 = \langle 1, x \rangle$ as invariant under $F[u]$, since

$$\begin{aligned} F[C_1 + C_2 x] &= A(C_1 + C_2 x)C_2 \\ &= b_1 + b_2 x \in W_2 \end{aligned}$$

where b_1 and b_2 are arbitrary constants given by

$$AC_1C_2 = b_1, \text{ and}$$
$$AC_2^2 = b_2.$$

This allows us to consider an exact solution of equation (5.1.1) as

$$u(x,t) = a_1(t) + a_2(t)x. \tag{5.4.2}$$

Substituting the value of $u(x,t)$ from equation (5.4.2) into the equation (5.1.1) and equating the coefficients of x^j, $j = 0, 1$, we get the following system of fractional differential equations

$$\frac{d^\alpha a_2(t)}{dt^\alpha} = (a_2(t))^2, \tag{5.4.3}$$

$$\frac{d^\alpha a_1(t)}{dt^\alpha} = Aa_1(t)a_2(t) \tag{5.4.4}$$

Eqns. (5.4.3-5.4.4) can be readily solved to yield

$$-\frac{1}{a_2(t)} = \frac{A}{\Gamma(1+\alpha)} \int dt^\alpha + s_1, \tag{5.4.5}$$

$$\log a_1(t) = \frac{A}{\Gamma(1+\alpha)} \int a_2(t) dt^\alpha + s_2 \tag{5.4.6}$$

and where s_1 and s_2 are arbitrary constants.
Using equation (5.4.2) and equations (5.4.5-5.4.6) one can easily obtain an exact solution of equation (5.1.1).

5.5 Discussion

In this chapter, an attempt has been made to illustrate the application of Lie symmetry approach to study time fractional Gardner equation. The similarity reductions and similarity solutions for the time fractional Gardner equation are presented. Some new analytical solutions are obtained by using the Lie group method of infinitesimals and some other exact solutions are obtained by the invariant subspace method. The Lie symmetry analysis shows that the underlying symmetry algebra of the time fractional Gardner equation is two dimensional.

The reduction of dimension in the symmetry algebra is due to the fact that the time fractional equations is not invariant under time translation symmetry. In Sec. 3, different similarity reductions are obtained. Using the Lie point symmetries, we have shown that the time fractional Gardner equation that is often very difficult to solve explicitly can be transformed into a nonlinear ODE of fractional order with Erdélyi-Kober fractional derivative which is not solvable as in the case of $\alpha = 1$. It would be appropriate to mention here that the fractional order significantly affects the properties of the equation. The main reason is that the fractional order $0 < \alpha < 1$ is an arbitrary parameter in the studied fractional model.

Chapter 6

Group Classification of Space-Time Fractional Coupled KdV Equation

6.1 Introduction

The KdV equation was derived by Diederik Johanes Korteweg and Gustav de Vries as a universal model to describe one-dimensional nonlinear waves in dispersive media without dissipation [70]. The existence of dispersion effect causes the spreading of the waveform, while the nonlinear effect causes the steepening of the waveform. Due to these two effects, a solitary wave is formed.

In this chapter, a space-time fractional coupled KdV equation with variable coefficients of the form:

$$u_t^{(\alpha)} + f(t)uu_x^{(\beta)} + g(t)vv_x^{(\beta)} + h(t)u_x^{(3\beta)} = 0,$$

$$v_t^{(\alpha)} + \delta(t)uv_x^{(\beta)} + k(t)v_x^{(3\beta)} = 0, \tag{6.1.1}$$

where $x \in (0, \infty)$, $t > 0$, $0 < \alpha, \beta < 1$, the coefficients $f(t), g(t), h(t), \delta(t)$ and

$k(t)$, are arbitrary smooth functions of variable t only, is examined for various types of explicit exact solutions. Here, $u_t^{(\alpha)}, v_t^{(\alpha)}$ and $u_x^{(\beta)}, v_x^{(\beta)}$ are the modified Riemann-Liouville derivatives with respect to time and space variables respectively

The chapter has been organized as follows. Section 6.2 is entirely devoted to showing how the symmetry group method can be used to generate various symmetries of the fractional coupled KdV system. In Section 6.3, we present the reduced systems of ordinary differential equations (ODEs) and their exact solutions. The final section contains the discussion and concluding remarks.

6.2 Symmetry Analysis of Space-Time Fractional Coupled KdV Equation

Lie group method of infitesimal transformations which essentially reduces the number of independent variables in PDE and reduces the order of ODE has been used widely in solving equations of mathematical physics. The classical Lie method for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformations and the associated determining equations are an over determined linear system.

In view of the algorithmic steps listed in section 1.5 of chapter 1, we proceed as follow.

Let us first assume that equation (6.1.1) admits the Lie symmetries of the form

$$\frac{\tilde{x^\beta}}{\Gamma(1+\beta)} = \frac{x^\beta}{\Gamma(1+\beta)} + \epsilon\xi(x,t,u,v) + o(\epsilon^2) \tag{6.2.1}$$

$$\frac{\tilde{t^\alpha}}{\Gamma(1+\alpha)} = \frac{t^\alpha}{\Gamma(1+\alpha)} + \epsilon\tau(x,t,u,v) + o(\epsilon^2) \tag{6.2.2}$$

$$\tilde{u} = u + \epsilon\eta^1(x,t,u,v) + o(\epsilon^2) \tag{6.2.3}$$

$$\tilde{v} = v + \epsilon\eta^2(x,t,u,v) + o(\epsilon^2) \tag{6.2.4}$$

where ϵ is the group parameter and ξ, τ and η^1, η^2 are the infinitesimals of the transformations for the independent and dependent variables, respectively. A group invariant solution of fractional coupled KdV equation(6.1.1) is a solution which can be mapped into another solution of equation (6.1.1) under the point transformations (6.2.1–6.2.4). The associated Lie algebra of infinitesimal symmetries of eqn. (6.1.1) is then the vector field of the form

$$V = \xi(x,t,u,v)\frac{\partial^\beta}{\partial x^\beta} + \tau(x,t,u,v)\frac{\partial^\alpha}{\partial t^\alpha} + \eta^1(x,t,u,v)\frac{\partial}{\partial u} + \eta^2(x,t,u,v)\frac{\partial}{\partial v}. \quad (6.2.5)$$

The fractional third order prolongation of (6.1.1) is given by

$$pr^{(3)}V = \xi(x,t,u)\frac{\partial^\beta}{\partial x^\beta} + \tau(x,t,u)\frac{\partial^\alpha}{\partial t^\alpha} + \eta^1(x,t,u)\frac{\partial}{\partial u} + \eta^{1(t)}\frac{\partial}{\partial u_t^{(\alpha)}} + \eta^{1(x)}\frac{\partial}{\partial u_x^{(\beta)}} +$$
$$\eta^{1(xx)}\frac{\partial}{\partial u_x^{(2\beta)}} + \eta^{1(xxt)}\frac{\partial}{\partial (u_t^{(\alpha)})_x^{(2\beta)}} + \eta^{1(xxx)}\frac{\partial}{\partial u_x^{(3\beta)}} + \eta^2(x,t,u)\frac{\partial}{\partial v} + \eta^{2(t)}\frac{\partial}{\partial v_t^{(\alpha)}} + \eta^{2(x)}\frac{\partial}{\partial v_x^{(\beta)}} +$$
$$\eta^{2(xx)}\frac{\partial}{\partial v_x^{(2\beta)}} + \eta^{2(xxt)}\frac{\partial}{\partial (v_t^{(\alpha)})_x^{(2\beta)}} + \eta^{2(xxx)}\frac{\partial}{\partial v_x^{(3\beta)}}. \quad (6.2.6)$$

Invoking the invariance as mentioned in step 2 of the method, the following relations from the coefficients of the first order of ϵ can be obtained:

where $\eta^{1(t)}, \eta^{2(t)}, \eta^{1(x)}, \eta^{2(x)}, \eta^{1(xxx)}$ and$\eta^{2(xxx)}$ are extended (prolonged) infinitesimals acting on an enlarged space that include all derivatives of dependent variables u and v with resepcet to the independent variables x and t. The next step requires finding the infinitesimals from the invariance conditions, by setting the coefficients of differentials equal to zero. It leads to a large number of PDEs in ξ, τ, η^1 and η^2 that need to be satisfied. Without going into the details of algebraic calculations, the set of determining equations for the group infinitesimals for the infinitesimals ξ, τ, η^1 and η^2 obtained, after equating the coefficients of various derivative terms to zero, is as follows:

$$\tau_x^{(\beta)} = \tau_u = \tau_v = 0 \quad (6.2.7)$$

$$\xi_u = \xi_v = \xi_x^{(2\beta)} = 0 \quad (6.2.8)$$

$$(\eta_u^1)_x^{(\beta)} = \eta_{uu}^1 = \eta_{uv}^1 = 0 \quad (6.2.9)$$

$$\eta^2_{uv} = \eta^2_{vv} = (\eta^2_u)^{(\beta)}_x = (\eta^2_v)^{(\beta)}_x = 0 \tag{6.2.10}$$

$$3h(t)\xi^{(\beta)}_x - (h(t)\tau)^{(\alpha)}_t = 0 \tag{6.2.11}$$

$$f(t)\eta^1 + f(t)u(\xi^{(\beta)}_x - \tau^{(\alpha)}_t) + g(t)v\eta^2_u - u\tau f(t)^{(\alpha)}_t + \xi^{(\alpha)}_t = 0 \tag{6.2.12}$$

$$g(t)\eta^2 + g(t)v\eta^2_v - g(t)v\eta^1_u - v(g(t)\tau)^{(\alpha)}_t = 0 \tag{6.2.13}$$

$$(\delta(t) - f(t))u\eta^2_u = 0 \tag{6.2.14}$$

$$\delta(t)\eta^1 + \delta(t)u\xi^{(\beta)}_x + \xi^{(\alpha)}_t - u(\delta(t)\tau)^{(\alpha)}_t - g(t)v\eta^2_u = 0 \tag{6.2.15}$$

$$(\eta^2)^{(\alpha)}_t + k(t)(\eta^2)^{(3\beta)}_x + \delta(t)u(\eta^2)^{(\beta)}_x = 0 \tag{6.2.16}$$

The general solution of equations (6.2.7–6.2.16) provides the following infinitesimal elements $\xi, \tau, \eta^1, \eta^2$; and the admissible forms of various coefficients in the system (6.1.1):

$$\xi = k_1 \frac{x^\beta}{\Gamma(1+\beta)} + k_5, \tag{6.2.17}$$

$$\tau = \frac{1}{H^{(\alpha)}_t}[3k_1 H + k_4] \tag{6.2.18}$$

$$\eta^1 = k_2 u, \tag{6.2.19}$$

$$\eta^2 = k_3 v, \tag{6.2.20}$$

where $k_1, ..., k_5$ are arbitrary constants and $k(t) = h(t) = H^{(\alpha)}_t$, with

$$(\xi f)^{(\alpha)}_t - (k_1 + k_2)f = 0, \tag{6.2.21}$$

$$(\xi g)^{(\alpha)}_t - (2k_3 + k_1 - k_2)g = 0, \tag{6.2.22}$$

$$(\xi \delta)^{(\alpha)}_t - (k_1 + k_2)\delta = 0. \tag{6.2.23}$$

The infinitesimal generators corresponding to Lie algebra of the fractional coupled KdV equation are given by

$$V_1 = \frac{3H}{H^{(\alpha)}_t}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{x^\beta}{\Gamma(1+\beta)}\frac{\partial^\beta}{\partial x^\beta}$$

$$V_2 = u\frac{\partial}{\partial u}$$

$$V_3 = v\frac{\partial}{\partial v} \tag{6.2.24}$$

$$V_4 = \frac{1}{H_t^{(\alpha)}} \frac{\partial^\alpha}{\partial t^\alpha}$$

$$V_5 = \frac{\partial^\beta}{\partial x^\beta}.$$

Further the set $\{V_1, V_2, V_3, V_4, V_5\}$forms a five dimensional Lie algebra under the Lie bracket $[X, Y] = XY - YX$. The commutator table for Lie algebra (6.2.24) is as follows:

Table 6.1: Commutator Table

	V_1	V_2	V_3	V_4	V_5
V_1	0	0	0	$-3V_4$	$-V_5$
V_2	0	0	0	0	0
V_3	0	0	0	0	0
V_4	$3V_4$	0	0	0	0
V_5	V_5	0	0	0	0

6.3 Some Exact Solutions of the Fractional Coupled KdV Equation

In the following, the similarity variable and the form of similarity solutions has been investigated. Further, the reduced system of ODEs for the fractional coupled KdV equation (6.1.1) are obtained and a number of explicit exact solutions to the system (6.1.1) are investigated. Using the infinitesimal generators (6.2.24) obtained in the previous section one can obtain a reduction of equations (6.1.1) to a system of ODEs after getting the similarity variable and the form by solving the fractional characteristic equations

$$\frac{\frac{(dt)^\alpha}{\Gamma(1+\alpha)}}{\tau} = \frac{\frac{(dx)^\beta}{\Gamma(1+\beta)}}{\xi} = \frac{du}{\eta^1} = \frac{dv}{\eta^2} \tag{6.3.1}$$

The reduced systems of ODEs corresponding to the following vector fields are examined:

(i) $V_1 + aV_2 + bV_3$

(ii) $V_2 + cV_3 + lV_4 + V_5$,

(iii) $V_2 + mV_3 + nV_4$,

(iv) $V_3 + rV_4 + V_5$

(v) $V_3 + sV_4$,

where a, b, c, l, m, n, r and s are arbitrary constant parameters. It is worth mentioning here that an attempt was made to drive a further system of ODEs of lower order through Lie group method, however, in almost all the cases, the symmetries obtained turned out to be the trivial ones. Therefore, we focus on attempting some particular types of explicit exact solutions for the reduced system of ODEs.

Vector field (i) For the infinitesimal generator

$$V_1 + aV_2 + bV_3 = \frac{3H}{H_t^{(\alpha)}}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{x^\beta}{\Gamma(1+\beta)}\frac{\partial^\beta}{\partial x^\beta} + au\frac{\partial}{\partial u} + bv\frac{\partial}{\partial v},$$

the similarity variable and the form of similarity solutions is obtained as follows: $X(x,t) = \frac{x^\beta}{\Gamma(1+\beta)}H(t)^{\frac{-1}{3}}$, $u(x,t) = \phi(X)H(t)^{\frac{-a}{3}}$ and $v(x,t) = \psi(X)H(t)^{\frac{-b}{3}}$, and the coefficients are given by the following relations:

$$f(t) = c_1 h(t)H(t)^{\frac{a-2}{3}}$$

$$g(t) = c_2 h(t)H(t)^{\frac{2b-a-2}{3}}$$

$$\delta(t) = c_3 h(t)H(t)^{\frac{a-2}{3}}.$$

This reduces the fractional coupled KdV equation to the following coupled nonlinear equations

$$\phi'''(X) - c_1\phi(X)\phi'(X) + c_2\psi(X)\psi'(X) - \frac{\zeta\phi'(X)}{3} - \frac{a\phi(X)}{3} = 0 \quad (6.15)$$

$$\psi'''(X) + c_3\phi(X)\psi'(X) - \frac{\zeta\psi'(X)}{3} - \frac{b\psi(X)}{3} = 0. \quad (6.16)$$

To solve the reduced system, we seek a special solution of the form

$$\phi(X) = A_0 + A_1X + A_2X^2 + A_3\frac{1}{X} + A_4\frac{1}{X^2},$$

$$\psi(X) = B_0 + B_1 X + B_2 X^2 + B_3 \frac{1}{X} + B_4 \frac{1}{X^2},$$

where A_i, B_i, $i = 0, 1, ..., 4$ are arbitrary constants.

Substituting these expressions for ϕ and ψ in the reduced system, we arrive at a system of algebraic equations, which on solving with the aid of Maple program provided an exact solution to the coupled equation (6.1.1)

$$u(x,t) = \frac{\frac{x^{3\beta}}{(\Gamma(1+\beta))^3} - 24H(t)}{2c_3 \frac{x^{2\beta}}{(\Gamma(1+\beta))^2} \sqrt{H(t)}} \tag{6.17}$$

$$v(x,t) = \frac{\frac{x^{3\beta}}{(\Gamma(1+\beta))^3} - 24H(t)}{2c_3 \frac{x^{2\beta}}{(\Gamma(1+\beta))^2}} \sqrt{\frac{c_3 - c_1}{2H(t)}} \tag{6.18}$$

Vector field (ii) In this case, the infinitesimal generator

$$V_2 + cV_3 + lV_4 + V_5 = u\frac{\partial}{\partial u} + cv\frac{\partial}{\partial v} + l\frac{1}{H_t^{(\alpha)}}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial^\beta}{\partial x^\beta},$$

gives the similarity variable and the form of similarity solution as:
$X(x,t) = \frac{H(t)}{l}$, $u(x,t) = \phi(X)e^{\frac{-H(t)}{l}}$ and $v(x,t) = \psi(X)e^{\frac{-cH(t)}{l}}$.
Where as, the form of the coefficient functions is given by

$$f(t) = c_4 h(t) e^{\frac{H(t)}{l}}$$

$$g(t) = c_5 h(t) e^{\frac{(2c-1)H(t)}{l}}$$

$$\delta(t) = c_6 h(t) e^{\frac{H(t)}{l}},$$

and the reduced form of the fractional coupled KdV equation is obtained as

$$\phi'''(X) + c_4\phi(X)\phi'(X) + c_5\psi(X)\psi'(X) - \frac{1}{l}\phi'(X) - \frac{1}{l}\phi(X) = 0 \tag{6.3.2}$$

$$\psi'''(X) + c_6\phi(X)\psi'(X) - \frac{1}{l}\psi'(X) - \frac{c}{l}\psi(X) = 0, \quad e \neq 0. \tag{6.3.3}$$

By using the same approach utilized in case (i) above to solve the system, it has the following exact solution

$$u(x,t) = (\frac{-c_5}{l}(1 + B_0 l) - \frac{c_5}{l(1 + c_4 c_5)}(\frac{H(t)}{l} - \frac{x^\beta}{\Gamma(1+\beta)}))e^{(\frac{-H(t)}{l})} \tag{6.3.4}$$

$$v(x,t) = (B_0 + \frac{1}{l(1+c_4c_5)}(\frac{H(t)}{l} - \frac{x^\beta}{\Gamma(1+\beta)}))e^{(\frac{H(t)}{(1+c_4c_5)l})} \tag{6.3.5}$$

Vector field (iii) Corresponding to the generator

$$V_2 + mV_3 + nV_4 = u\frac{\partial}{\partial u} + mv\frac{\partial}{\partial v} + \frac{n}{H_t^{(\alpha)}}\frac{\partial^\alpha}{\partial t^\alpha},$$

we get the following form of similarity variable and similarity solution
$\zeta(x,t) = \frac{x^\beta}{\Gamma(1+\beta)}$, $\quad u(x,t) = F(\zeta)e^{\frac{-H(t)}{n}}$ and $\quad v(x,t) = G(\zeta)e^{\frac{-mH(t)}{n}}$.
The coefficient functions in this case are related by the following relations

$$f(t) = c_7h(t)e^{\frac{H(t)}{n}}$$

$$g(t) = c_8h(t)e^{\frac{(2m-1)H(t)}{n}}$$

$$\delta(t) = c_9h(t)e^{\frac{H(t)}{n}},$$

and the reduced form of fractional coupled KdV equation is obtained as

$$P'''(X) + c_7P(X)P'(X) + c_8Q(X)Q'(X) - \frac{P(X)}{n} = 0 \tag{6.3.6}$$

$$Q'''(X) + c_8P(X)Q'(X) - \frac{mG(X)}{n} = 0 \tag{6.3.7}$$

which gives the solution

$$u(x,t) = (A_0 + \frac{m}{c_9n}\frac{x^\beta}{\Gamma(1+\beta)})e^{-\frac{H(t)}{n}} \tag{6.3.8}$$

$$v(x,t) = \frac{\sqrt{\frac{m(c_9-c_7m)}{c_8}}}{c_9mn}(A_0 + \frac{m}{c_9n}\frac{x^\beta}{\Gamma(1+\beta)})e^{-\frac{mH(t)}{n}} \tag{6.3.9}$$

Vector field (iv) The infinitesimal generator

$$V_3 + rV_4 + V_5 = v\frac{\partial}{\partial v} + r\frac{1}{H_t^{(\alpha)}}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\partial^\beta}{\partial x^\beta}$$

introduces the folowing similarity variable and similarity solution
$X(x,t) = \frac{H(t)}{r} - \frac{x^\beta}{\Gamma(1+\beta)}$, $\quad u(x,t) = \phi(X)$ and $\quad v(x,t) = \psi(X)e^{\frac{-H(t)}{r}}$, $r \neq 0$,
and the form of the coefficient functions in this case is given by

$$f(t) = c_{10}h(t)$$

$$g(t) = c_{11}h(t)e^{\frac{2H(t)}{r}}$$

$$\delta(t) = c_{12}h(t)$$

Here, the reduced form of fractional coupled KdV equation is achieved as

$$\phi'''(X) + c_{11}\psi(X)\psi'(X) + c_{10}\phi(X)\phi'(X) - \frac{1}{r}\phi'(X) = 0 \tag{6.3.10}$$

$$\psi'''(X) + c_{12}\phi(X)\psi'(X) - \frac{1}{r}\psi'(X) - \frac{1}{r}\psi(X) = 0, \quad r \neq 0. \tag{6.3.11}$$

This yields the following solution to the fractional coupled KdV equation (6.1.1)

$$u(x,t) = \frac{1}{2c_{11}} + 2e^{(-\frac{H(t)+4\frac{x^\beta}{\Gamma(1+\beta)}}{8})} \tag{6.3.12}$$

$$v(x,t) = 2e^{(\frac{H(t)-4\frac{x^\beta}{\Gamma(1+\beta)}}{8})}. \tag{6.3.13}$$

Vector field (v) The infinitesimal generator

$$V_3 + sV_4 = v\frac{\partial}{\partial v} + s\frac{1}{H_t^{(\alpha)}}\frac{\partial^\alpha}{\partial t^\alpha}$$

introduces the folowing similarity variable and similarity solution
$X(x,t) = \frac{x^\beta}{\Gamma(1+\beta)}$, $u(x,t) = \phi(X)$ and $v(x,t) = \psi(X)e^{\frac{-H(t)}{s}}$, $s \neq 0$,
and the form of the coefficient functions in this case is given by

$$f(t) = c_{13}h(t)$$

$$g(t) = c_{14}h(t)e^{\frac{2H(t)}{s}}$$

$$\delta(t) = c_{15}h(t)$$

The reduced form of fractional coupled KdV equation under this case is

$$\phi'''(X) + c_{14}\psi(X)\psi'(X) + c_{13}\phi(X)\phi'(X) = 0 \tag{6.3.14}$$

$$\psi'''(X) + c_{15}\phi(X)\psi'(X) - \frac{1}{s}\psi(X) = 0, \quad s \neq 0. \tag{6.3.15}$$

which gives the solution

$$u(x,t) = c_0 + \frac{\frac{x^\beta}{\Gamma(1+\beta)}}{sc_{15}} \tag{6.3.16}$$

$$v(x,t) = c_0 + \frac{\frac{x^\beta}{\Gamma(1+\beta)}}{sc_{15}} e^{\left(\frac{-H(t)}{s}\right)}, \tag{6.3.17}$$

where c_0 is an arbitrary constant and $c_{14} = -c_{13}$.

Further the one parameter Lie group of point transformations corresponding to each infinitesimal generator can be used to generate more solutions of the fractional coupled KdV equation.

6.4 Discussion

Symmetry properties and reductions of a fractional coupled KdV system with variable coefficients is presented using the method of Lie group of infinitesimal transformations. The infinitesimals of the group of transformations which leaves the fractional coupled KdV system invariant and the admissible forms of the coefficients are obtained. Corresponding to various linear combinations of the infinitesimal generators, it is shown that the fractional coupled KdV system reduces to coupled nonlinear ordinary differential equations in each case, which is further studied with the aim of deriving certain exact solutions.

Chapter 7

Conclusions

The importance of the fractional order partial differential equations due to their recent occurrence in the study of many processes in science and engineering and also the various limitations posed by the integer order derivative models, have been the prime reasons for making the study put up in the thesis entitled "Symmetry Analysis of Some Fractional Order Partial Differential Equations". The study of symmetries and exact solutions of nonlinear partial differential equations has great theoretical and practical importance. These exact solutions for nonlinear systems are used as models for physical or numerical investigations and often replicate qualitatively on the behaviour of more complicated solutions. More specifically, the thesis deals with nonlinear partial differential equations of fractional order representing some interesting physical systems which are–the space-time fractional Burgers-Poisson equation, time fractional potential Burgers' equation, variable coefficient space-time fractional potential Burgers' equation, time fractional Gardner and space-time fractional coupled KdV equation, from the view point of their underlying Lie symmetries of infinitesimal transformations.

The main purpose of Lie symmetry method is to reduce PDEs to ODEs by introducing suitable similarity variables. Here, similarity analysis has been successfully performed on various nonlinear fractional order partial differential equations. To determine the admissible symmetries two methods- one based on non-differentiable functions and the other one based on differentiable functions, have been utilized. It has been illustrated that the fractional partial differential equations possess similarity solutions, exactly as its counterparts with integer-order derivatives for the first approach, while in the second approach the fractional differential equations possess fewer dimensional Lie algebra than the integer one. In both cases by using conveniently defined similarity variables the fractional equations reduce to ordinary differential equations which are further solved for some group invariant solutions. After obtaining the point symmetries of the system under investigation, the attempt has been to reduce the number of independent variables of the system and then reduced equations have been further studied by several methods including invariant subspace method. It may be noted that the solutions obtained for various systems in the thesis are completely new, which have never been reported before. These solutions can further be used as a supportive tool in designing and testing of numerical algorithms.

In chapters 2, 3 and 5, Lie group method has been applied on equations with constant coefficients, whereas, in chapters 4 and 6 equations with variable coefficients have been studied. In case of variable coefficients equations, most of the solutions involve an arbitrary coefficient function which enables us to control and discuss the behaviour of solutions as governed by the choice of this arbitrary function. Also, the Lie group method has been applied on a coupled system of fractional differential equations.

In short, we can say that work in the thesis is devoted to investigating a range of applications of continous symmetry groups to physically important systems of fractional partial differential equations. Finally, it is worth mentioning that in spite of the focus on the exact solutions, the author found it really difficult at times to find symmetries for fractional differential equations as symmetries for integer order differential equations can be obtained by some mathematical software. Keeping in view this limitation, it will be really interesting if such software can be developed. It has also been an extremely difficult task to handle the reduced system of ODEs for extracting the exact solutions. In some cases, the solutions obtained are of very specific nature and further application of Lie group method on the reduced system led only to trivial symmetries. Thus, the general solution of reduced ODEs, their physical interpretation and the study of higher order symmetries of fractional order differential equations bring forth tremendous scope for future work.

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